A Theory of Conditional Sets

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Abstract

In this thesis, we develop a theory of conditional sets. Conditional set theory is sufficiently rich in order to allow for a conditional mathematical reasoning, the possibility of which we demonstrate by constructing a conditional general topology and a conditional real analysis. We prove the conditional version of the following theorems: Ultrafilter Lemma, Tychonoff, Borel-Lebesgue, Heine-Borel, Bolzano-Weierstraß, and Debreu's Gap Lemma. Moreover, we prove the conditional version of those results in classical mathematics which are needed in the proofs of these theorems, starting from set theory. We discuss the connection of conditional set theory to sheaf, topos and L^0-theory.

Zusammenfassung

Diese Arbeit befasst sich mit der Entwicklung einer Theorie bedingter Mengen. Bedingte Mengenlehre ist reich genug um einen bedingten mathematischen Diskurs zu führen, dessen Möglichkeit wir durch die Konstruktion einer bedingten Topologielehre und bedingter reeller Analysis aufzeigen. Wir beweisen die bedingte Version folgender Sätze: Ultrafilterlemma, Tychonoff, Borel-Lebesgue, Heine-Borel, Bolzano-Weierstraß, und das Gaplemma von Debreu. Darüberhinaus beweisen wir die bedingte Version derjenigen Resultate der klassischen Mathematik, die in den Beweisen dieser Sätze benötigt werden, beginnend mit der Mengenlehre. Wir diskutieren die Verbindung von bedingter Mengenlehre zur Garben-, Topos- und L^0-Theorie.

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Contents

Two Kinds of Intelligence

There are two kinds of intelligence: one acquired,
as a child in school memorizes facts and concepts
from books and from what the teacher says,
collecting information from the traditional sciences
as well as from the new sciences.

With such intelligence you rise in the world.
You get ranked ahead or behind others
in regard to your competence in retaining
information. You stroll with this intelligence
in and out of fields of knowledge, getting always more
marks on your preserving tablets.

There is another kind of tablet, one
already completed and preserved inside you.
A spring overflowing its springbox. A freshness
in the center of the chest. This other intelligence
does not turn yellow or stagnate. It's fluid,
and it doesn't move from outside to inside
through conduits of plumbing-learning.

This second knowing is a fountainhead
from within you, moving out.

Rumi *(translation by Coleman Barks)*

1 Introduction

The simple idea behind conditional set theory is: A statement may hold locally, although it fails to be true globally. We illustrate what that implies with the help of the following examples. Let $([0,1], \mathfrak{B}, \lambda)$ be the Lebesgue probability space and L^0 be the collection of all equivalence classes of real random variables. An open or closed L^0-interval is a set of the form

$$(X,Y) := \{Z \in L^0 : X < Z < Y\} \quad \text{or} \quad [X,Y] := \{Z \in L^0 : X \leq Z \leq Y\},$$

where $X, Y \in L^0$ with $X < Y$, and the inequalities are to be understood in the λ - almost everywhere sense. Let further $\mathfrak{T}$ be the topology generated by the collection of all L^0-open intervals and $A \in \mathfrak{B}$ be such that $0 < \lambda(A) < 1$. We make the following observations:

- The intersection of

$$V := 1_A[0,1] + 1_{A^c}[2,3] \quad \text{and} \quad W := 1_A[0,1] + 1_{A^c}[4,5]$$

  is empty, although V and W coincide on some event of positive probability. That is, V and W intersect conditionally. The result of this conditional intersection lives on the event A and occurs with probability $\lambda(A)$.

- Neither of the random variables

$$X := 1_A \frac{1}{2} + 1_{A^c}4 \quad \text{and} \quad Y := 4$$

  is an element of V. However, while $1_A X$ is an element of $1_A V$, one cannot find any $B \in \mathfrak{B}$ with positive measure such that $1_B Y$ is a member of $1_B V$. In this case, X is *conditionally* an element of V, and Y is not.

- The sequence

$$X_n := 1_A 5 + 1_{A^c} n, \qquad n \in \mathbb{N},$$

  does not converge in $\mathfrak{T}$. However, (X_n) conditioned on any event $B \subseteq A$ converges in the conditional topology $1_B\mathfrak{T}$. In other words, (X_n) converges *conditionally* with positive probability.

- The family of L^0-open intervals

$$O_1 := \left(X_1(\omega) := -\frac{1}{2} + \frac{1}{2}\omega, Y_1(\omega) := -\frac{1}{2} + \frac{3}{2}\omega\right),$$
$$O_n := \left(X_n(\omega) := -\frac{1}{2} + 3 \cdot 2^{n-2}\omega, Y_n(\omega) := -\frac{1}{2} + 3 \cdot 2^{n-1}\omega\right), \quad n \geq 2,$$

  is a conditional open covering of the closed interval $[0,1]$. This means that for every random variable $Z \in [0,1]$ there exists a partition $(A_n) \subseteq \mathfrak{B}$ such that $Z = \sum 1_{A_n} X_n$ where $X_n \in O_{i_n}$ for every $n \in \mathbb{N}$. It is not possible to find a finite subfamily of (O_n) which conditionally covers $[0,1]$. In the classical sense, $[0,1]$ is not compact. However, observe that there exists a partition

$$B_n := \left(\frac{1}{2^n}, \frac{1}{2^{n-1}}\right], \quad n \in \mathbb{N},$$

  such that on each B_n we find a finite subfamily of (O_n) conditionally covering $1_{B_n}[0,1]$, namely the family $\{O_1, \ldots O_{n+1}\}$.

The previous examples shed light on a mathematical reasoning which is different from the common one in the following aspects:

- there is an effect from an underlying source of conditions (here from $\mathfrak{B}$) which acts locally on objects that allow for conditioning (here by 1_A);
- it involves operations on sets which differ from classical set operations;
- it stresses some local property which is not a global one;
- a global property is negated if it holds nowhere locally;
- a global property is accepted to hold if it already holds on each part of a partition.

Conditional set theory is a language to express conditionality in a consistent way by specifying sets for which a conditional reasoning can be carried out and by defining suitable operations on these sets which allow for a mathematical discourse.

We think that the structure best capturing the idea of a meaningful collection of conditions is a Boolean algebra. A Boolean algebra $(\mathcal{A}, \wedge, \vee, {}^c, 0, 1)$ is a set with three operations and two distinguished elements satisfying certain well-known axioms. One can interpret the first operation as the coincidence of two conditions, the second as their disjunction, and the third one as the expression of the absence of some condition. The distinguished element 0 represents the condition which never occurs (the empty condition) and 1 the condition which always occurs (the global condition). Boolean algebras are ubiquitous in mathematics: measure spaces, measure algebras, algebras of projections, regular open algebras, clopen spaces, and Lindenbaum-Tarski algebras. We consider Boolean algebras which are complete and satisfy the additional assumption **(P)** which for every family $(a_i) \subseteq \mathcal{A}$ requires that there is a partition $(b_j) \subseteq \mathcal{A}$ such that for all b_j there exists some a_{i_j} with $b_j \wedge a_{i_j} = b_j$. Let $\mathscr{A}$ be the class of all such Boolean algebras.

A conditional set X is a structure which allows for an action from some $\mathcal{A} \in \mathscr{A}$. Formally, X is a family of sets (X_a) and a family of surjective functions (γ_a) parametrized by the elements of some $\mathcal{A} \in \mathscr{A}$. Each set X_a represents the elements of the global set X_1 conditioned on a while $\gamma_a : X_1 \to X_a$ describes the process of conditioning. This process is required to satisfy consistency and stability. Consistency means that, if the occurrence of a condition a implies the occurrence of another condition b, that is $a \leq b$, then passing from X_b to X_a should make the following diagram commute

$$\begin{tikzcd} X_1 \arrow[r] \arrow[dr] & X_b \arrow[d] \\ & X_a. \end{tikzcd}$$

Stability on the other hand means that for every partition $(a_i) \subseteq \mathcal{A}$ and every family (x_i), where $x_i \in X_{a_i}$ for each i, there exists $x \in X_1$ which conditioned on a_i coincides with x_i for all i. The first assumption ensures that all structures built upon conditional sets and statements about them given globally can be equally considered locally (or conditionally). The second assumption is the first one's counterpart, allowing to paste localized objects and statements to a global one.

The structure of a conditional set is localizable by restricting it to some relative algebra $\mathcal{A}_a$ of the Boolean algebra $\mathcal{A}$ on which it is defined. A conditional set X defined on a Boolean algebra $\mathcal{A}$ is conditionally included in a conditional set Y on another Boolean algebra $\mathcal{B}$, if $\mathcal{A}$ is a relative algebra of $\mathcal{B}$ and Y restricted to $\mathcal{A}$ includes the structure of X. This means that each set $X_a \subseteq Y_a$ and the conditioning functions of X are restrictions of the ones of Y. As in the classical set theory, the conditional inclusion is the key to define the other basic operations: conditional power set, intersection, union, complement, product, relation and function. Probably the most fundamental result in conditional set theory is that the conditional power set together with the conditional set operations forms a complete Boolean algebra. Classical set theory is a particular case of conditional set theory. More precisely, classical set theory is conditional set theory on the trivial algebra $\{0, 1\}$.

A conditional subset of the conditional product of two conditional sets is a conditional relation. Conditional functions are functional conditional relations. The conditional power set of the conditional power set of a conditional set is likely a place for conditional topologies and conditional filters. On conditional sets we develop the following concepts and prove the following theorems:

- conditional set theory, number systems and cardinality;
- conditional continuity, filters, convergence, compactness;
- the conditional version of the Ultrafilter Lemma and of Tychonoff's Theorem;
- construction of the conditional real numbers and its algebraic and topological characterization;

- basis of real analysis, the conditional version of the Bolzano-Weierstraß, the Heine-Borel and the Borel-Lebesgue Theorems.

When one works with conditional sets there are some recurring principles which we list in the following:

- The duality between local and global: Since a conditional set admits an action from some $\mathcal{A} \in \mathscr{A}$ satisfying consistency and stability, every conditional structure admits an action from some $\mathcal{A} \in \mathscr{A}$ satisfying consistency and stability.
- If a property does not hold globally, but locally on some condition, then it is always possible to construct the greatest condition on which it is true. This is the essence of assumption **(P)**.
- A property of a conditional set X is expressed by the respective classical property on X_1 with additional requirements. For systems of conditional sets (e.g. conditional topologies and filters), the connection between classical and conditional concepts is realized via a suitable base. The connection to classical concepts is a consequence of the surjectivity of the function part of conditional sets.
- A property of a conditional set X which is based on a logical affirmation is proven on X_1. In contrast, a property which relys on a logical negation is conditionally true if and only if the only condition on which it holds is 0.

Conditional set theory in form of L^0-theory has found numerous applications in mathematical finance, see [CHKP11, FKV12, CH11, CS12]. We give an application to decision theory. A classical preference relation is represented numerically by the real numbers. Debreu's Gap Lemma is the technical tool to prove the existence of continuous representations. Conditional preferences are conditional relations which are conditionally complete and transitive. A conditional preference is a model of a decision process where a preference exists among choices only if some additional assumptions are realized. The conditional real numbers may represent numerically conditional preferences. To this end, we prove the conditional version of Debreu's Gap Lemma.

Connection to the literature

To the best of our knowledge the notion of a conditional set does not exist in the previous literature, however, it is closely related to the notion of a sheaf on a complete Boolean algebra [LM92] on the one hand, and to Boolean-valued sets [Sco67, Sol70, Bel05] on the other hand.

Ordinary set theory is the most prominent example of local set theories. From Cohen's forcing arose in the work of Scott [Sco67] and Solovay [Sol70] the Boolean-valued models of the Zermelo-Fraenkel set theory with choice (ZFC). Every set can be identified uniquely with the characteristic functions of its subsets which is a Boolean-valued function taking its values in the trivial algebra $\{0,1\}$. Given any complete Boolean algebra $\mathcal{A}$, one obtains the universe of $\mathcal{A}$-valued sets by replacing $\{0,1\}$ by $\mathcal{A}$ and considering functions taking values in $\mathcal{A}$. One constructs the universe of $\mathcal{A}$-valued sets in analogy to the construction of the von Neumann universe by a recursion on ordinals. The first-order language of the universe of $\mathcal{A}$-valued sets is the extension of the first-order language of set theory. The truth value of each $\mathcal{A}$-sentence is an element of $\mathcal{A}$. A recent summary of Boolean-valued models can be found in [Bel05]. A Boolean-valued universe is a non-standard model of ZFC. Applications of non-standard models evolve from the comparison of a structure in the standard model with its realization in a non-standard model. For example, a function space in a standard model replaces the real numbers in a non-standard model. Boolean-valued models found applications in vector-valued optimization, see [KK99]. Since it was shown in [Hig73] that the category of $\mathcal{A}$-valued sets, where $\mathcal{A}$ is a complete Boolean algebra, is equivalent to the topos of sheaves on $\mathcal{A}$, we restrict the discussion to the connection of conditional sets with the topos of sheaves.

An elementary topos is a generalized universe of sets in which it is possible to have a mathematical discourse. It is a category in which the fundamental relations between objects obey to the same laws as in the category of sets. It is known that category theory may serve as an alternative foundation of mathematics different from axiomatic set theory. The difference is partly epistemological. In axiomatic set theory the world is given by the universe of sets and the membership relation. Properties of sets are formulated intrinsically as properties of independent entities. In contrast, in the category of sets the world is given by a collection of objects and arrows. The properties of an object are expressed by its relations to other objects. This points to a interdependent point of view which implies that as soon as a collection of objects and arrows satisfies certain basic relations, it serves as a model for mathematics, independent of the specific form of the objects and arrows which may differ significantly from sets and set functions. In the language of topos theory this reads as follows. Every topos which is Boolean, well-pointed, satisfies the axiom of choice, and has a natural numbers object is equiconsistent with a weak version of ZFC, see [LM92, VI.10]. Not every topos is Boolean, well-pointed or has a natural numbers object. In any topos, those results in ordinary mathematics hold true the proofs of which are finitist and constructive. In a topos with a natural numbers object one can remove the restriction that the proof is finitist, and if the topos is Boolean, the restriction that the proof is constructive.

A sheaf is a structure which encodes local-global information with respect to some mathematical structure. Originally, a sheaf was defined on the collection of all open subsets of a topological space, see [Ten75] for an introduction. In [AGV72] its definition was generalized, to make it possible to define a sheaf on some suitable small category $\mathcal{C}$. To this end, so-called Grothendieck topologies were introduced. A Boolean algebra $\mathcal{A} \in \mathscr{A}$ is such a small category and a standard Grothendieck topology on $\mathcal{A}$ is the sup-topology. The sup-topology consists for each $a \in \mathcal{A}$, of families of elements of $\mathcal{A}$ smaller than a whose supremum is a. The collection of partitions of elements of a complete Boolean algebra $\mathcal{A}$ satisfying **(P)** forms a Grothendieck basis for the sup-topology on $\mathcal{A}$. Sheaves for the sup-topology whose arrow parts consist of surjective functions can be identified with conditional sets. There is a difference between the inclusion relation realized in the Grothendieck topos of sheaves and the conditional inclusion introduced on the class of all conditional sets. Hence, the conditional set operations differ from the local set operations on sheaves. The requirement that the function part of a conditional set consists of surjections together with the Assumption **(P)** simplifies the structure of the corresponding sheaf. It is thus easier to work with conditional sets than with sheaves or Boolean-valued sets whose structure is not immediately tangible to a working mathematician.

Conditional set theory developed from the study of L^0-modules, where L^0 is the set of equivalence classes of real-valued measurable functions on a σ-finite measure space. The theory of L^0-modules was initiated in [FKV09]. Similar ideas date as far back as 1942 when Menger introduced a probabilistic metric in [Men42] which specifies the distance between two points by a probability distribution rather than a number, see for a comprehensive account of the theory and its applications [SS05]. A source of richness of the theory is the choice of a triangular function on the set of probability distributions to identify the triangular inequality for a probabilistic metric. In a random metric space the distance between two points is defined by a random variable on a previously fixed probability space rather than by its distribution. To analyze topological properties of probabilistic metric spaces the (ε, λ)-topology was introduced: Two points are close to each other if their distance is small with high probability. Among the topological properties which are studied are continuity, completion, compactness, and fixed point theorems, see [HP01, SS05, She71] and the references therein.

A probabilistic metric space endowed with an (ε, λ)-topology is usually not a locally convex topological vector space, and thus does not immediately admit convex analysis, [Gou10]. Guo replaces in a random metric space a random variable by its equivalence class in L^0, which leads to the structure of a random locally convex module. This makes it possible to study the conjugate space of a random metric space, [GS11, Gou10, Gou11]. On the set of equivalence classes of random variables on some probability space the (ε, λ)-topology coincides with the topology of convergence in probability.

Motivated by financial applications, Filipović, Kupper and Vogelpoth initiated independently the theory of L^0-modules in [FKV09, KV08]. In their work an L^0-module is understood as a conditional extension of a real vector space. The idea is to replace $\mathbb{R}$ consequently by L^0. This is possible due to the rich structure of L^0: It is a complete lattice-ordered associative algebra over $\mathbb{R}$. The main interest was to establish a conditional duality theory which could be applied to conditional risk measures. Later Cheridito, Kupper and Vogelpoth [CKV12] expanded the scope of the theory of L^0-modules by translating analytic and algebraic results from $\mathbb{R}^d$ to $(L^0)^d$. The strategy in L^0-theory is to lift the algebraic and topological structure of a real vector space to an L^0-module. Hence, the L^0-topology (the interval topology $\mathfrak{T}$ defined at the beginning of this introduction) is constructed in analogy to the topology of the real line. The proofs of the conditional theorems are in line with the proofs of the respective unconditional ones. The L^0-approach differs in some aspects from probabilistic analysis. L^0-theory is applicable as soon as there exists a σ-finite measure. The results obtained are measurable by construction and do not rely on measurable selection arguments.

In his survey article [Gou10], Gou compares the duality results obtained for L^0-modules with respect to the L^0-topology with the ones obtained with respect to the (ε, λ)-

topology. He shows that results concerning topological completion are equivalent, however, the L^0-topology allows to prove stronger separation results. As shown in [Guo08], it is generally not possible to establish compactness results for the (ε, λ)-topology analogous to the results which hold in locally convex topological vector spaces, for example the Banach-Alaoglu Theorem and the Krein-Šmulian Theorem. In [DJKK13], we show that the concept of conditional compactness is the appropriate concept to establish these results for L^0-modules.

Conditional compactness is a natural consequence of the definition of a conditional topology and conditional topology is a natural consequence of conditional set theory. This connection was previously not available in the L^0-theory. Conditional set theory together with its notion of conditional real numbers and its calculus is a significant generalization of the theory of L^0-modules. The space L^0 can be understood as a conditional set on the measure algebra $\mathcal{A}$ associated to a σ-finite measure space and it is shown that L^0 with the L^0-topology is conditionally homeomorphic to the conditional real line on $\mathcal{A}$. This is a strong theoretical justification of the methods previously applied in L^0-theory and the intuitive anticipation that L^0 is a real numbers object. See for a similar discussion Scott's "Boolean Models and Nonstandard Analysis" in [LoT69]. Conditional set theory with its concept of conditional real numbers is the appropriate language to systematically studying L^0-modules, or to say it in the language of conditional set theory, to study conditional real vector spaces. All results previously obtained for L^0-modules [FKV09, KV08, CKV12, DKKS13] are equally results about conditional real vector spaces, due to the aforementioned identification.

There is an accumulation of results on conditional sets, ranging from set theory and general topology to real and functional analysis. It is natural to ask in how far results from classical mathematics can be transferred to conditional sets and if so, does there exist a transfer principle. For Boolean-valued models such a transfer principle exists for real analysis, see [KK99]. In an attempt to answer this question, we tried to formalize conditional sets in the language of category theory. Recall that in topos theory it is shown that certain elementary toposes, for example the topos of sheaves on a complete Boolean algebra, serve as a model of a restricted version of ZFC. To this end, it is necessary to show that the topos is Boolean, has a natural numbers object and satisfies the axiom of choice. In order to formalize conditional set theory, primarily developed in the von Neumann-Gödel-Barneys set theory, in the language of category theory, it is necessary to describe the fundamental operations, e.g. the conditional inclusion and the conditional Cartesian product, by means of conditional functions. We construct a category representing the objects of conditional set theory, obtained by a factorization on the product category of a fixed $\mathcal{A} \in \mathscr{A}$ with the subcategory of sheaves on $\mathcal{A}$ whose function part consists of surjections. Remarkably, this category shares many properties of a Boolean topos with a natural numbers object and the axiom of choice, but unfortunately not all. The question for the logical limits of conditional set theory and a transfer principle still remains open.

This thesis is organized as follows. In Chapter 3, we define the elements of conditional set theory. These are conditional sets, the conditional inclusion, conditional products, power sets, relations, functions, numbers, families and cardinality. We prove theorems which are essential for basic mathematical reasoning on conditional sets, the most important of which being the Boolean character of the conditional power set. In Chapter 4, we introduce the basic concepts of conditional topology. This includes conditional topological spaces, bases, continuity, filters, convergence, and compactness. We prove theorems which are necessary to speak about closeness in a conditional framework. In Chapter 5, we construct the conditional real numbers, characterize its usual conditional topology, apply it to define conditional metric spaces and to prove the conditional version of Debreu's Gap Lemma. Finally, in Chapter 6 we discuss the connection of conditional sets to sheaves and topos theory, construct the category of conditional sets and show that it is not an elementary topos. In the Appendix 1 and 2, we give an introduction into Boolean algebras and category theory, sufficient to follow the discussion in this thesis.

The Chapters 3–5 are based on two joint works [DJKK13] and [DJ13] with Samuel Drapeau, Martin Karliczek and Michael Kupper.

2 Notations

This thesis is carried out in the von Neumann-Gödel-Barneys set theory (NGB), a conservative extension of Zermelo-Fraenkel set theory with choice (ZFC), which allows to deal with large collections, so-called proper classes. That implies that NGB is an extension of the language of ZFC and a statement in ZFC is provable in NGB if and only if it can already be shown in ZFC.

The natural numbers are $\mathbb{N} = \{1, 2, \ldots\}$ and we use the standard notation $\{*\}$ for a one-point set. For a family of sets $(X_i)_{i\in I}$, by $\coprod_{i\in I} X_i$ we denote its disjoint union. Relevant definitions and results from the theory of Boolean algebras and from category theory are collected in Appendix 1 and 2, respectively.

Let $\mathcal{A} = (\mathcal{A}, \wedge, \vee, {}^c, 0, 1)$ be a Boolean algebra. Recall that the relation $a \leq b$ if $a \wedge b = a$ defines a distributive complemented lattice. For any family $(a_i)_{i\in I}$ in $\mathcal{A}$, we denote by $\vee a_i = \vee_{i\in I} a_i$ and $\wedge a_i = \wedge_{i\in I} a_i$ its supremum and infimum, respectively. For $a \in \mathcal{A}$, the relative algebra of $\mathcal{A}$ with respect to a is denoted by $\mathcal{A}_a$. For two families $(a_i)_{i\in I}$ and $(b_j)_{j\in J}$ in $\mathcal{A}$, we define $(a_i)_{i\in I} \wedge (b_j)_{j\in J} := (a_j \wedge b_j)_{i\in I, j\in J}$. We introduce the notations $\Delta = \Delta(\mathcal{A}) := \{(a, b) \in \mathcal{A}^2 : a \leq b\}$ and $\Delta_a := \Delta(\mathcal{A}_a)$. A partition of an element $a \in \mathcal{A}$ is a family $(a_i)_{i\in I}$ in $\mathcal{A}$ such that $\vee a_i = a$ and $a_i \wedge a_j = 0$ if $i \neq j$. We denote by $\mathcal{K}(a)$ the set of all partitions of a. A partition of unity is an element of $\mathcal{K}(1)$. Notice that we allow 0 to be a member of a partition.

3 Conditional Set Theory

In Section 3.1, we define a conditional set on some complete Boolean algebra. The complete Boolean algebras under consideration satisfy some additional assumption which guarantees that one can construct the greatest condition for which a statement about conditional sets is true. In Section 3.2, we introduce the conditional inclusion relation on the class of all conditional sets which leads us to the definition of a conditional power set. We prove the main result of this chapter: That the conditional power set has the structure of a complete Boolean algebra. A conditional binary relation is a conditional subset of the conditional product of two conditional sets, a particular example of which is a conditional function, which we introduce in Section 3.3. Finally, in Section 3.4, we define conditional families which are systems of sets parametrized by a conditional set and will turn out to be useful for defining mathematical structures on conditional sets and prove statements about them.

3.1 Sets

We consider as a source of conditions a complete Boolean algebra $\mathcal{A}$ satisfying the following additional assumption which is essential for the theory of conditional sets:[1]

(P) For every family $(a_i)_{i\in I}$ in $\mathcal{A}$ there exists a partition $(b_j)_{j\in J} \in \mathcal{K}(\vee a_i)$ such that for all $j \in J$ there is $i_j \in I$ with $b_j \leq a_{i_j}$.

Denote by $\mathscr{A}$ the class of all complete Boolean algebras which satisfy **(P)**.

Examples 3.1.1. (i) Every atomic and complete Boolean algebra is in $\mathscr{A}$. For example, the power set algebra $\mathcal{P}(X)$ of any set X, where the atoms are the singletons $\{x\}$ for each $x \in X$.

(ii) Every σ-complete Boolean algebra which satisfies the countable chain condition is in $\mathscr{A}$. Completeness of $\mathcal{A}$ is due to [Tar37]. In order to verify **(P)**, let (c_i) be in $\mathcal{A}$. Then [GH09, Chapter 30, Lemma 1] implies the existence of a countable subfamily (b_n) of (c_i) which has the same set of upper bounds as (c_i). Define $a_1 := b_1$ and $b_n := a_n \wedge (\vee_{k=1}^{n-1} a_k)^c$, for $n \geq 2$. Then (a_n) is a partition of $\vee c_i$ satisfying **(P)**.

For instance, the associated measure algebra of a σ-finite measure space is σ-complete Boolean algebra satisfying the countable chain condition, see [MKB89, Chapter 22].

[1] The definition of a conditional set, a conditional power set, and of a conditional product does not depend on Assumption **(P)**, neither the results up to Lemma 3.2.10. All other results strongly rely on it.

(iii) The next example is taken from [MKB89, Chapter 1.8]. Let

$$H = \{(x_n)_{n\in\mathbb{N}} : x_n \in \mathbb{R}, \sum_{n\in\mathbb{N}} x_n^2 < \infty\}$$

be the Hilbert space l^2. For $N \subseteq \mathbb{N}$, define $e_N : H \to H$ by $e_N(x) = x \cdot 1_N$, where the multiplication is understood pointwise and 1_N is the indicator function of N. Then e_N is a projection of H. Moreover, $\mathcal{A} := \{e_N : N \subseteq \mathbb{N}\}$ is a Boolean algebra, where

$$\begin{aligned} e \vee f &:= e + f - e \circ f, \\ e \wedge f &:= e \circ f, \\ e^c &:= 1_H - e. \end{aligned}$$

The Boolean algebra $\mathcal{A}$ is completely isomorphic to $\mathcal{P}(\mathbb{N})$, and thus an element of $\mathscr{A}$. ◊

We define a conditional set.

Definition 3.1.2. Let $\mathcal{A} \in \mathscr{A}$, $(X_a)_{a\in\mathcal{A}}$ be a family of sets, and $(\gamma_a)_{a\in\mathcal{A}}$ be a family of surjective functions $\gamma_a : X_1 \to X_a$. The structure

$$X := (X_a, \gamma_a)_{a\in\mathcal{A}}$$

is a *conditional set* if and only if

(i) X_0 is a singleton;

(ii) *(identity)* γ_1 is the identity;

(iii) *(consistency)* $\gamma_a(x) = \gamma_a(y)$, whenever $\gamma_b(x) = \gamma_b(y)$, $x, y \in X_1$, and $(a, b) \in \Delta$;

(iv) *($\mathcal{A}$-stability)* for every partition of unity $(a_i)_{i\in I}$ and for every $(x_i)_{i\in I} \in \prod_{i\in I} X_{a_i}$ there exists a unique $x \in X_1$ such that $\gamma_{a_i}(x) = x_i$ for all $i \in I$.

We identify two conditional sets X and Y if the only difference is $X_0 \neq Y_0$.[2] Therefore, there exists only one conditional set on the degenerate algebra $\mathcal{A} = \{0\}$ which is denoted by $\mathbf{0}$ and which is called the *conditional empty set*.

Examples 3.1.3. (i) Let $\mathcal{A}$ be the trivial algebra $\{0, 1\}$. Then every non-empty set X can be turned into a conditional set by setting $X_1 := X$ and $X_0 := \{*\}$.

(ii) Every one-point set $\{*\}$ defines a conditional set on every $\mathcal{A} \in \mathscr{A}$ by setting $X_a := \{*\}$ and $\gamma_a := \mathrm{id}$ for all $a \in \mathcal{A}$.

(iii) Every $\mathcal{A} \in \mathscr{A}$ is a conditional set X on itself by defining $X_a := \mathcal{A}_a$ and $\gamma_a(b) := a \wedge b$ for each $a \in \mathcal{A}$. ◊

[2] Note that this relation is an equivalence relation on the proper class of all conditional sets.

Let $X = (X_a, \gamma_a)_{a\in\mathcal{A}}$ be a conditional set. The surjectivity of γ_a and $X_0 \neq \emptyset$ imply that $X_a \neq \emptyset$ for every $a \in \mathcal{A}$. Furthermore, γ_0 is uniquely determined since there is only one function from X_1 to a singleton set. For every $(a, b) \in \Delta$, surjectivity and consistency guarantee the existence of a unique function $\gamma_a^b : X_b \to X_a$ such that the following diagram commutes:

$$\begin{tikzcd} X_1 \arrow[r, "\gamma_b"] \arrow[dr, "\gamma_a"'] & X_b \arrow[d, "\gamma_a^b"] \\ & X_a. \end{tikzcd} \tag{3.1.1}$$

Indeed, define $\gamma_a^b(x) := \gamma_a(\tilde{x})$ where $\tilde{x} \in X_1$ is such that $\gamma_b(\tilde{x}) = x$.

The following proposition shows that a conditional set is equivalent to a richer structure which we identify with a sheaf in Chapter 6.

Proposition 3.1.4. *Let $X = (X_a, \gamma_a)_{a\in\mathcal{A}}$ be a conditional set. Then it holds*

(i) γ_a^a is the identity for every $a \in \mathcal{A}$;

(ii) $\gamma_a^b \circ \gamma_b^c = \gamma_a^c$ for every $a \leq b \leq c$;

(iii) for all $a \in \mathcal{A}$, every $(a_i)_{i\in I} \in \mathcal{K}(a)$, and each $(x_i)_{i\in I} \in \prod_{i\in I} X_{a_i}$ there exists a unique $x \in X_a$ such that $\gamma_{a_i}^a(x) = x_i$ for all $i \in I$.

Conversely, let $\mathcal{A} \in \mathscr{A}$, $(X_a)_{a\in\mathcal{A}}$ be a family of sets and $(\gamma_a^b)_{(a,b)\in\Delta}$ be a family of surjections $\gamma_a^b : X_b \to X_a$ fulfilling (i), (ii) and (iii). Then $X = (X_a, \gamma_a^1)_{a\in\mathcal{A}}$ is a conditional set.

Proof. Let $X = (X_a, \gamma_a)_{a\in\mathcal{A}}$ be a conditional set. By the commutative diagram (3.1.1), (i) and (ii) are satisfied. As for (iii), let $z \in X_{b^c}$. By $\mathcal{A}$-stability, there exists a unique $\tilde{x} \in X_1$ such that $\gamma_{a_i}(\tilde{x}) = x_i$ for all $i \in I$ and $\gamma_{a^c}(\tilde{x}) = z$. Define $x := \gamma_a(\tilde{x}) \in X_a$. Let $y \in X_a$ be such that $\gamma_{a_i}^a(y) = x_i$ for each $i \in I$. By $\mathcal{A}$-stability, there exists a unique $\tilde{y} \in X_1$ such that $\gamma_{a_i}(\tilde{y}) = x_i$ for all $i \in I$ and $\gamma_{a^c}(\tilde{y}) = z$. Uniqueness yields $\tilde{x} = \tilde{y}$, and thus $x = y$, due to consistency.

Conversely, if $(X_a)_{a\in\mathcal{A}}$ is a family of sets and $(\gamma_a^b)_{(a,b)\in\Delta}$ a family of surjections fulfilling (i), (ii), and (iii), then $X = (X_a, \gamma_a^1)_{a\in\mathcal{A}}$ satisfies identity, consistency, and $\mathcal{A}$-stability. The empty family is a partition of 0, since $\vee_\emptyset = 0$, and it holds $\prod_\emptyset = \{\emptyset\}$. Consequently, X_0 consists of exactly one element, by (iii). $\square$

In [MKB89] and [GH09], a partition is defined as a family of positive pairwise disjoint elements. We allow 0 to be an element of a partition. Assuming that a partition does not contain 0, yields a definition of a conditional set equivalent to the Definition 3.1.2. Indeed, let $(x, x_0) \in X_1 \times X_0$ for the partition of unity $(1, 0)$ and $x \in X_1$ for the partition of unity 1. Since X_0 is a singleton, x is the unique element satisfying $\mathcal{A}$-stability for both families. The reason why we stipulate that a partition may contain 0 is that it bears some technical advantages.

We introduce the following notations for the sake of simplicity. Let $X = (X_a, \gamma_a)_{a\in\mathcal{A}}$ be a conditional set. By abuse of language, we define a function

$$\begin{aligned} a : \coprod_{b\in\mathcal{A}} X_b &\longrightarrow \coprod_{c\in\mathcal{A}_a} X_c \\ x \in X_b &\longmapsto ax := \gamma^b_{a\wedge b}(x) \in X_{a\wedge b}, \end{aligned}$$

for every $a \in \mathcal{A}$ and call it the *conditioning* on a. Given $(a_i) \in \mathcal{K}(a)$ for some $a \in \mathcal{A}$, a family

$$(x_i) = (x_i)_{i\in I} \in \prod X_{b_i}$$

is called *matching* for (a_i) if $a_i \leq b_i$ for all $i \in I$. The unique element $x \in X_a$ satisfying $a_i x = a_i x_i$ for all $i \in I$ is the *amalgamation* of (x_i). We denote by

$$[a_i, x_i] = [a_i, x_i]_{i\in I} \subseteq \mathcal{A}_a \times X$$

the combination of a partition $(a_i) \in \mathcal{K}(a)$ and a matching family (x_i) for (a_i). Given $[a_i, x_i] \subseteq \mathcal{A}_a \times X$, we write

$$\sum a_i x_i = \sum_{i\in I} a_i x_i$$

for the amalgamation of (x_i). If $I = \{1, \dots, n\}$, we also write $a_1x_1 + \ldots + a_nx_n$.

Lemma 3.1.5. *Let X be a conditional set.*

(i) For every $a, b \in \mathcal{A}$ and all $[b_i, x_i] \subseteq \mathcal{A}_b \times X$, it holds

$$a \sum b_i x_i = \sum (a \wedge b_i) x_i \in X_{a\wedge b}.$$

(ii) For every $a \in \mathcal{A}$, all $(a_i)_{i\in I} \in \mathcal{K}(a)$, and every $[b_{ij}, x_{ij}]_{j\in J_i} \subseteq \mathcal{A}_{b_i} \times X$ where $a_i \leq b_i$ for each $i \in I$, it holds

$$\sum_{i\in I} a_i \sum_{j\in J_i} b_{ij} x_{ij} = \sum_{j\in J_i,\, i\in I} (a_i \wedge b_{ij}) x_{ij} \in X_a. \tag{3.1.2}$$

Proof. As for the first assertion, let $x = \sum_{i\in I} a_i x_i$ and $y = \sum_{i\in I} (a \wedge b_i) x_i$. Then

$$(a \wedge b_i) ax = a(b_i x) = a(b_i x_i) = (a \wedge b_i) x_i,$$

for every $i \in I$. Since $(a \wedge b_i)_{i\in I}$ is a partition of $a \wedge b$, it follows from Proposition 3.1.4 that $ax = y$. The second assertion follows analogously by conditioning both sides of (3.1.2) to $a_i \wedge b_{ij}$ for all $j \in J_i$ and $i \in I$ since $(x_{ij})_{j\in J_i, i\in I}$ is a matching family for $(a_i \wedge b_{ij})_{j\in J^i, i\in I} \in \mathcal{K}(a)$. □

Next, we show how to generate a conditional set from some arbitrary set with respect to some $\mathcal{A} \in \mathscr{A}$. Let E be a non-empty set and denote by $\sum_{i\in I} a_i x_i := (a_i, x_i)_{i\in I}$ for any

$(a_i, x_i)_{i\in I} \subseteq \mathcal{A} \times E$ for which (a_i) is a partition of some $a \in \mathcal{A}$. For every $a \in \mathcal{A}$, define

$$E_a := \left\{ \sum a_i x_i : (a_i, x_i)_{i\in I} \subseteq \mathcal{A} \times E \right\},$$

where two families $\sum a_i x_i$ and $\sum b_j y_j$ are identified if

$$\vee\{a_i : x_i = z\} = \vee\{b_j : y_j = z\}, \quad \text{for all } z \in E.$$

Let $\gamma_a : E_1 \to E_a$ be given by $\sum a_i x_i \mapsto \sum (a \wedge a_i) x_i$. Then $(E_a, \gamma_a)_{a\in\mathcal{A}}$ is a conditional set. Indeed, for every $\sum a_i x_i \in E_a$, it holds $\gamma_a(\sum a_i x_i + a^c y) = \sum a_i x_i$ for any $y \in E$. Thus, γ_a is surjective. If $(a_i) \in \mathcal{K}(a)$ and $b \in \mathcal{A}$, then $(a_i \wedge b) \in \mathcal{K}(a \wedge b)$. This implies identity and consistency. Since $\vee_\emptyset = 0$, it holds $E_0 = \{\emptyset\}$. As for the $\mathcal{A}$-stability, let $(a_i) \in \mathcal{K}(1)$ and $(\sum_j b_{ij} x_{ij})_{i\in I} \in \prod E_{a_i}$. For every $i \in I$, it holds

$$\gamma_{a_i}\left(\sum_{i,j} b_{ij} x_{ij} \right) = \sum_j b_{ij} x_{ij},$$

and $\sum b_{ij} x_{ij}$ is unique since $(b_{ij}) \in \mathcal{K}(1)$.

Definition 3.1.6. Let $\mathcal{A} \in \mathscr{A}$ and E be a non-empty set. We call

$$\mathbf{E} := (E_a, \gamma_a)_{a\in\mathcal{A}}$$

the conditional set *generated* by E with respect to $\mathcal{A}$.

Examples 3.1.7. For $E = \mathbb{N}$, $\mathbb{Z}$ or $\mathbb{Q}$, we call $\mathbf{N}$ the *conditional natural numbers*, $\mathbf{Z}$ the *conditional integers*, and $\mathbf{Q}$ the *conditional rational numbers* with respect to $\mathcal{A}$, respectively. $\Diamond$

We will often rely on the following remark.

Remark 3.1.8. The structure of a conditional set $X = (X_a, \gamma_a)_{a\in\mathcal{A}}$ is representable by X_1 which is due to the surjectivity of the γ_a. Most properties of conditional sets can be expressed and proved in terms of properties of X_1. For notational convenience, if there exists no risk of confusion, we will identify X with X_1. We will often compare a classical concept with its conditional counterpart, for example a filter with a conditional filter or a topology with a conditional topology. Whenever we talk of a classical concept on a conditional set X we omit the word "conditional" and we clearly mean a structure on X_1. For instance, given a conditional set X, a filter $\mathfrak{F}$ on the *set* X is a classical filter on X_1. We deal in the same way with conditional and classical functions. For example, when we say $f : X \to Y$ is a conditional function, we mean a conditional function of conditional sets, however, when we say $f : X \to Y$ is a function, then we mean the classical function $f_1 : X_1 \to Y_1$ of sets. $\blacklozenge$

3.2 Inclusion

Definition 3.2.1. Let $X = (X_a, \gamma_a)_{a \in \mathcal{A}}$ be a conditional set and $a \in \mathcal{A}$. A non-empty subset $Y \subseteq X_a$ is $\mathcal{A}_a$*-stable* if $\sum a_i x_i \in Y$ for all $[a_i, x_i] \subseteq \mathcal{A}_a \times X$ such that $x_i \in Y$ for all $i \in I$. To each $\mathcal{A}_a$-stable subset Y is *associated* a conditional set $Y := (Y_b, \delta_b)_{b \in \mathcal{A}_a}$, where $Y_b := \gamma_b^a(Y)$ and δ_b is the restriction of γ_b^a to Y_b for every $b \in \mathcal{A}_a$. In this case we say that the conditional set Y *lives* on a.

For any non-empty $Y \subseteq X_a$,

$$\left\{ \sum a_i x_i : [a_i, x_i] \subseteq \mathcal{A}_a \times X, (x_i) \subseteq Y \right\}$$

is an $\mathcal{A}_a$-stable subset of X_a, due to Lemma 3.1.5. The associated conditional set is called the $\mathcal{A}$*-stable hull* of Y, and we denote it by $cond(Y)$.

Examples 3.2.2. Let $X = (X_a, \gamma_a)_{a \in \mathcal{A}}$ be a conditional set.

(i) By Proposition 3.1.4, X_a is $\mathcal{A}_a$-stable for each $a \in \mathcal{A}$. The associated conditional set is called the *restriction* of X to a, and is denoted by aX.

(ii) Every $x \in X_a$ for some $a \in \mathcal{A}$ is $\mathcal{A}_a$-stable. The conditional set associated to a singleton is called a *conditional singleton.*

(iii) Every one-point set is an $\mathcal{A}_0$-stable subset of any conditional set X, and the conditional set which is associated to it is $\mathbf{0}$. ♢

Remark 3.2.3. Let X and Y be two non-empty sets such that $X \subseteq Y$. Then $\mathbf{X} \sqsubseteq \mathbf{Y}$ where $\mathbf{X}$ and $\mathbf{Y}$ are the conditional sets generated by X and Y, respectively. ♦

Definition 3.2.4. Let $X = (X_a, \gamma_a)_{a \in \mathcal{A}}$ and $Y = (Y_b, \delta_b)_{b \in \mathcal{B}}$ be two conditional sets. We say that Y is a *conditional subset* of X, and write $Y \sqsubseteq X$, if and only if $\mathcal{B} = \mathcal{A}_a$ for some $a \in \mathcal{A}$, $Y_a \subseteq X_a$ is $\mathcal{A}_a$-stable, and Y is the conditional set associated to Y_a.

Proposition 3.2.5. *The conditional inclusion relation $\sqsubseteq$ is a partial order on the class of all conditional sets.*

Proof. Let $Z = (Z_c, \rho_c)_{c \in \mathcal{C}}$, $Y = (Y_b, \delta_b)_{b \in \mathcal{B}}$ and $X = (X_a, \gamma_a)_{a \in \mathcal{A}}$ be three conditional sets. The reflexivity of $\sqsubseteq$ is immediate. As for the antisymmetry, assume that $Y \sqsubseteq X$ and $X \sqsubseteq Y$. It follows that $\mathcal{A} = \mathcal{B}$ and $Y_1 = X_1$, and thus $X = Y$. As for the transitivity, assume that $Z \sqsubseteq Y$ and $Y \sqsubseteq X$. It follows that $\mathcal{B} = \mathcal{A}_a$ for some $a \in \mathcal{A}$, and $\mathcal{C} = \mathcal{B}_b$ for some $b \in \mathcal{B}$. Hence, $\mathcal{C} = \mathcal{A}_b$, since $b \leq a$. Moreover, it holds $Z_b \subseteq Y_b$ and $Y_a \subseteq X_a$. By consistency, $Y_c \subseteq X_c$ for all $c \leq a$, and thus $Z_b \subseteq X_b$. Therefore, ρ_c is the restriction of δ_c^b on Z_b for all $c \leq b$, and δ_c the restriction of γ_c^a on Y_a for all $c \leq b$. Since δ_c^b is the restriction of γ_c^b on Y_b, it follows that ρ_c is the restriction of γ_c^b on Z_b for all $c \leq b$. Thus, $Z \sqsubseteq X$. □

Proposition 3.2.6. *Let $X = (X_a, \gamma_a)_{a\in\mathcal{A}}$ be a conditional set. Then*

$$\mathcal{S}(X) = (\mathcal{S}_a, \delta_a)_{a\in\mathcal{A}}$$

is a conditional set where

$$\mathcal{S}_a := \{Y : Y \text{ is a conditional set associated to some } \mathcal{A}_a\text{-stable subset of } X_a\}$$

and $\delta_a : \mathcal{S}_1 \to \mathcal{S}_a$ is given by $Y \mapsto aY$.

Proof. The function δ_1 is the identity and $\mathcal{S}_0$ consists of only $\mathbf{0}$. For surjectivity, let $Y \in \mathcal{S}_a$ for some $a \in \mathcal{A}$. Then $Z := \{ay + a^c x : y \in Y, x \in X\}$ is an $\mathcal{A}$-stable subset of X_1, due to Lemma 3.1.5. By construction, $aZ = Y$. Consistency can be shown by a similar argument. As for the $\mathcal{A}$-stability, let (a_i) be a partition of unity, $(Y^i) \in \prod \mathcal{S}_{a_i}$, and define $Z := \{\sum a_i y_i : y_i \in Y^i, i \in I\}$. By Lemma 3.1.5, it follows that $Z \subseteq X_1$ is $\mathcal{A}$-stable. By construction, $a_i Z = Y^i$ for all $i \in I$. Its uniqueness follows from the uniqueness of amalgamations in X. □

Note that $a\mathcal{S}(X) = \mathcal{S}(aX)$.

Proposition 3.2.7. *Let $X = (X_a, \gamma_a)_{a\in\mathcal{A}}$ be a conditional set. Then*

$$\mathcal{P}(X) = (\mathcal{P}_a, \delta_a)_{a\in\mathcal{A}}$$

is a conditional set where

$$\mathcal{P}_a := \{bY : Y \in \mathcal{S}(aX), b \in \mathcal{A}_a\}$$

and $\delta_a : \mathcal{P}_1 \to \mathcal{P}_a$ is given by $bY \mapsto (a \wedge b)Y$.

Proof. Clearly, $\mathcal{P}_0 = \mathbf{0}$ and $\mathcal{P}_a \subseteq \mathcal{P}_1$, and thus $\delta_a(Y) = Y$ for all $Y \in \mathcal{P}_a$. Hence, δ_a is surjective for all $a \in \mathcal{A}$, δ_1 is the identity, and it follows consistency. As for the $\mathcal{A}$-stability, let $(a_i) \in \mathcal{K}(1)$ and $(Y^i) \in \prod \mathcal{P}_{a_i}$. Then Y_i lives on some $b_i \leq a_i$ for every i. Therefore, $b_i \wedge b_j = 0$ for all $i, j \in I$ with $i \neq j$. Let $b := \vee b_i$ and define

$$Z := \left\{\sum b_i y_i : y_i \in b_i Y^i, i \in I\right\}.$$

By Lemma 3.1.5, Z is an $\mathcal{A}_b$-stable subset of X_b, and thus the conditional set Z is an element of $\mathcal{P}_1$. Furthermore, $\delta_{a_i}(Z) = Y^i$ for all $i \in I$ and its uniqueness follows from the Proposition 3.2.6. □

Definition 3.2.8. The conditional set $\mathcal{P}(X)$ is called the *conditional power set* of X.

Since the conditional power set is a conditional set, it is possible to define the conditional power set of the conditional power set of a conditional set. Observe that $a\mathcal{P}(X) = \mathcal{P}(aX)$ and $\mathcal{S}(X) \sqsubseteq \mathcal{P}(X)$.

Examples 3.2.9. (i) The conditional power set of $\mathbf{0}$ is $\mathbf{0}$.

(ii) Let $\mathcal{A} = \{0,1\}$ and X be a non-empty set. Then $\mathcal{P}(X)$ is isomorphic to the power set of X if we send $\mathbf{0}$ to the empty set. $\Diamond$

The following lemma is an important consequence of Assumption **(P)**.

Lemma 3.2.10. *Let X be a conditional set and $Y, Z \in \mathcal{P}(X)$. Then*

$$A = \{a \in \mathcal{A} : aY \sqsubseteq Z\}$$

has a greatest element.

Proof. Denote by $a^* = \vee A$ and pick $(b_i) \in \mathcal{K}(a^*)$ fulfilling **(P)**. For every $i \in I$, it holds $b_i Y \sqsubseteq a_i Y \sqsubseteq Z$, where $a_i \in A$ is such that $b_i \leq a_i$. The $\mathcal{A}$-stability of $\mathcal{P}(X)$ yields

$$a^* Y = \sum b_i Y \sqsubseteq Z. \qquad \square$$

Theorem 3.2.11. *The structure $(\mathcal{P}(X), \sqcap, \sqcup, {}^{\sqsubset}, \mathbf{0}, X)$ is a complete Boolean algebra for every conditional set X, where*

(i) $\sqcup_{i\in I} Y^i$ is the conditional set associated to

$$\left\{\sum b_j y_j : (b_j) \in \mathcal{K}(\vee a_i), y_j \in Y^{i_j}, b_j \leq a_{i_j}\right\};$$

(ii) $\sqcap_{i\in I} Y^i$ is the conditional set associated to

$$\bigcap_{i\in I} a_* Y^i \quad \text{and} \quad a_* := \vee\{a \in \mathcal{A} : a \leq \wedge a_i, \cap a Y^i \neq \emptyset\};$$

(iii) $Y^{\sqsubset} := \sqcup\{Z \in \mathcal{P}(X) : Y \sqcap Z = \mathbf{0}\}$

for $Y \in \mathcal{P}(X)$ and $(Y^i)_{i\in I} \subseteq \mathcal{P}(X)$, where Y^i lives on a_i for every $i \in I$.

Proof. It suffices to verify that $(\mathcal{P}(X), \sqsubseteq)$ is a complete complemented distributive lattice with least element $\mathbf{0}$ and greatest element X. Let $(Y^i)_{i\in I}$ be a non-empty family in $\mathcal{P}(X)$ where Y^i lives on a_i for each $i \in I$, $Y^k \in \mathcal{P}(X)$ where Y^k lives on a_k for $k = 1,2,3$ and $Y \in \mathcal{P}(X)$.

(Step 1) We prove that $Z = \sqcup Y^i$ is the least upper bound of (Y^i). We define $a^* := \vee a_i$, and observe that Z is an $\mathcal{A}_{a^*}$-stable subset of X_{a^*}, due to Lemma 3.1.5. Let $\sum b_j y_j \in Z$ and pick $y_i \in Y^i$ for some $i \in I$ arbitrarily. Then

$$a_i y_i + (a_i^c \wedge a^*) \sum b_j y_j \in Z,$$

since the family consisting of a_i and $(a_i^c \wedge a^* \wedge b_j)$ is an element of $\mathcal{K}(a^*)$ and since each of the members of this family is smaller than some a_i. Hence,

$$Y^i \sqsubseteq a_i Z \sqsubseteq Z, \quad \text{for every } i \in I,$$

since y_i was chosen arbitrarily. Thus, Z is an upper bound. Let $W \in \mathcal{P}(X)$ be such that $Y^i \sqsubseteq W$ for all $i \in I$. This implies that W lives on some $b \geq a^*$. Moreover, $Y^i \sqsubseteq a^*W$ for every $i \in I$. Without loss of generality, assume that $a^*W = W$. For $\sum b_j y_j \in Z$, it holds

$$b_j \sum b_j y_j = b_j y_j \in b_j Y^{i_j} \subseteq b_j W, \quad \text{for all } j.$$

Due to the $\mathcal{A}$-stability of W, it holds $\sum b_j y_j \subseteq a^*W$, implying $a^*Z \sqsubseteq a^*W$. Thus, $Z \sqsubseteq W$. Therefore, Z is the least upper bound of (Y^i).

(Step 2) We prove that $Z = \sqcap Y^i$ is the greatest lower bound of (Y^i). Note that Z is an $\mathcal{A}_{a_*}$-stable subset of X_{a_*} since a_* is attained, due to the $\mathcal{A}$-stability of X. Since $\cap a_* Y^i \sqsubseteq Y^i$ for all $i \in I$, the conditional set Z is a lower bound. Let $W \in \mathcal{P}(X)$ be such that $W \sqsubseteq Y^i$ for every $i \in I$. Then $W \sqsubseteq bY^i$ for all $i \in I$, where W lives on b. Therefore, $W \subseteq \cap bY^i$ which implies $b \leq a_*$. For any $z \in Z$, it holds

$$W \sqsubseteq bW + (a_* \wedge b^c)z \sqsubseteq Z.$$

Hence, $W \sqsubseteq Z$, and thus Z is the greatest lower bound of the (Y^i).

(Step 3) We prove the distributive law

$$(Y^1 \sqcap Y^2) \sqcup (Y^1 \sqcap Y^3) = Y^1 \sqcap (Y^2 \sqcup Y^3).$$

The conditional inclusion from left to right follows from the fact that $(\mathcal{P}(X), \sqsubseteq)$ is a lattice, due to the previous two steps and Proposition 3.2.5. For the reverse conditional inclusion, let

$$x \in Y^1 \sqcap (Y^2 \sqcup Y^3) = a_* Y^1 \cap a_*(Y^2 \sqcup Y^3),$$

where $a_* \leq (a_1 \wedge a_2) \vee (a_1 \wedge a_3)$. Then there exists $y_1 \in Y^1$ such that $x = a_* y_1$. Moreover, $x = a_*(b_2 y_2 + b_3 y_3)$ where $b_k \leq a_k$ and $y_k \in Y^k$ for $k = 2, 3$ and $b_2 \vee b_3 = a_2 \vee a_3$. This implies

$$b_k x = (a_* \wedge b_k) y_k \in (a_* \wedge b_k) Y^1 \cap (a_* \wedge b_k) Y^k \sqsubseteq Y^1 \sqcap Y^k, \quad k = 2, 3.$$

Therefore,

$$x = a_*(b_2 y_2 + b_3 y_3) \in Z^2 \sqcup Z^3 \sqsubseteq (Y^1 \sqcap Y^2) \sqcup (Y^1 \sqcap Y^3),$$

where $Z^k := (a_* \wedge b_k) Y^1 \cap (a_* \wedge b_k) Y^k$ for $k = 2, 3$.

(Step 4) Finally, we prove complementation. Clearly, $\mathbf{0}$ and X are the least and greatest element of $\mathcal{P}(X)$. Hence, $\mathbf{0} \sqsubseteq Y \sqcap Y^{\sqsubset} \sqsubseteq Y \sqcup Y^{\sqsubset} \sqsubseteq X$. Suppose, for the sake of

contradiction, that $Y \sqcap Y^{\sqsubset}$ lives on some $a > 0$. However, it holds then $Y \sqcap Y^{\sqsubset} \sqsubseteq Y^{\sqsubset}$ and $(Y \sqcap Y^{\sqsubset}) \sqcap Y = Y \sqcap Y^{\sqsubset} \neq \mathbf{0}$ which is contradictory. Conversely, let $x \in X$ and $b = \vee\{a \in \mathcal{A} : ax \sqsubseteq Y\}$. By Lemma 3.2.10, $bx \sqsubseteq Y$ and $b^c x \sqcap Y = \mathbf{0}$. Hence, $b^c x \sqsubseteq Y^{\sqsubset}$, and thus $x \in Y \sqcup Y^{\sqsubset}$. □

Definition 3.2.12. We call the operations $\sqcup$, $\sqcap$ and $^{\sqsubset}$ the *conditional union*, the *conditional intersection* and the *conditional complement*, respectively. By convention, we set $\sqcap_{\emptyset} := X$ and $\sqcup_{\emptyset} := \mathbf{0}$.

The conditional complement of $Y \sqsubseteq X$ can also be characterized as the collection of all $x \in X$ such that $ax \notin aY$ for all $a > 0$ or, equivalently, if $ax \in aY$ only if $a = 0$. The following rules hold true in $\mathcal{P}(X)$, by [MKB89, Chapter 1, Lemma 1.33]. For non-empty families $(Y^i)_{i\in I}$ and $(Y^{ij})_{i\in I,\, j\in J}$ in $\mathcal{P}(X)$ and $Z \in \mathcal{P}$, it holds

(i) (*de Morgan's law*) $(\sqcup Y^i)^{\sqsubset} = \sqcap (Y^i)^{\sqsubset}$;

(ii) (*distributivity*) if I is finite, $\sqcap_{i\in I} \sqcup_{j\in J} Y^{ij} = \sqcup \left\{ \sqcap_{i\in I} Y^{if(i)} : f : I \to J \right\}$;

(iii) (*associativity*) $\sqcup_i(\sqcup_j Y^{ij}) = \sqcup_{ij} Y^{ij}$.

Remark 3.2.13. In general, stronger distributive laws may fail, since due to the construction of the conditional intersection and union, the distributive law of the conditional power set is equivalent to the distributive law of the underlying $\mathcal{A} \in \mathscr{A}$. ♦

Proposition 3.2.14. *Let $X = (X_a, \gamma_a)_{a\in\mathcal{A}}$ be a conditional set. The conditional power set $\mathcal{P}(X)$ is atomic if and only if $\mathcal{A}$ is so.*

Proof. Let A be the set of atoms of $\mathcal{A}$. Then the conditional singletons bx, where $b \in A$ and $x \in X$, constitute the set of atoms of $\mathcal{P}(X)$. Conversely, if A is atomless, then for each $a > 0$ there exists $0 < b < a$ such that $bx \sqsubseteq ax$ and $bx \neq ax$. This implies that $\mathcal{P}(X)$ is atomless. □

Remark 3.2.15. Let $Y, Z \in \mathcal{S}(X)$ and $Y^1, Y^2 \in \mathcal{P}(X)$ for some conditional set X. Then

(i) $Y \sqcap Z = Y \cap Z$, whenever $Y \sqcap Z \in \mathcal{S}(X)$;

(ii) $Y \sqsubseteq Z$ implies $Y \subseteq Z$;

(iii) $Y \sqcup Z = cond(Y \cup Z)$;

(iv) $Y^1 \sqcup Y^2 = b_1 Y^1 + b_2(Y^1 \sqcup Y^2) + b_3 Y^2$, where $b_1 = a_1 \wedge a_2^c$, $b_2 = a_1 \wedge a_2$, $b_3 = a_2 \wedge a_1^c$ and Y^i lives on a_i for each $i = 1, 2$. ♦

3.3 Products, relations and functions

Definition 3.3.1. Let $\mathcal{A} \in \mathscr{A}$ and $(X^i)_{i\in I}$ be a non-empty family of conditional sets, where X^i is a conditional set on $\mathcal{A}_{a_i}$ for some $a_i \in \mathcal{A}$ and each $i \in I$. The *conditional Cartesian product* of the family (X^i) is

$$\prod X^i := \Big(\prod X^i_a, (\gamma^i_a)\Big)_{a\in\mathcal{A}_{\wedge a_i}}.$$

By convention, we set $\prod_\emptyset := \mathbf{0}$.

Remark 3.3.2. Let X and Y be two non-empty sets, and $\mathbf{X}$, $\mathbf{Y}$ and $\mathbf{X}\times\mathbf{Y}$ be the conditional sets generated by X, Y, and $X\times Y$ with respect to some $\mathcal{A}\in\mathscr{A}$, respectively. Since conditional products are evaluated pointwise, the conditional product of $\mathbf{X}$ and $\mathbf{Y}$ equals $\mathbf{X}\times\mathbf{Y}$. ♦

Let X and Y be two conditional sets on some fixed non-degenerate $\mathcal{A}\in\mathscr{A}$.

Definition 3.3.3. A *conditional binary relation* R between X and Y is a conditional set $R \sqsubseteq X\times Y$. A pair $(x,y)\in X\times Y$ is said to be in relation conditioned on $b\in\mathcal{A}$ if $(bx,by)\in R_b$, and we write xR_by. Let $R\sqsubseteq X\times X$ be a conditional binary relation on X where R lives on a. We say that R is conditionally *reflexive, symmetric, antisymmetric*, or *transitive* if every R_b is reflexive, symmetric, antisymmetric, or transitive, respectively, for all $b\leq a$.

Let $R\sqsubseteq X\times Y$ be a conditional binary relation, where R lives on $a\in\mathcal{A}$. By means of Lemma 3.2.10, there exists a greatest condition $b\in\mathcal{A}$ such that xR_by for every pair $(x,y)\in X\times Y$. In particular, every pair (x,y) is in relation conditioned on 0, and there exists no pair (x,y) which is in relation on some condition b such that $b\wedge a^c>0$. Every conditional binary relation R is given by a family of binary relations $R_b\subseteq X_b\times Y_b$ parametrized by $b\in\mathcal{A}_a$, if R lives on a. Moreover, it holds

$$\sum b_i(x_i,y_i) = \Big(\sum b_i x_i, \sum b_i y_i\Big)\in R_b$$

for some $b\leq a$ and $(b_i)\in\mathcal{K}(b)$, whenever $x_iR_{b_i}y_i$ for each $i\in I$. Conversely, every $\mathcal{A}_a$-stable subset of $X_a\times Y_a$ for some $a\in\mathcal{A}$ is associated to a conditional binary relation. Without loss of generality, we assume henceforth that conditional relations live on 1.

Given a conditional relation $R\sqsubseteq X\times X$ on some conditional set X, note that if R_1 is symmetric or reflexive, then R_a is so for all $a\in\mathcal{A}$. The properties of antisymmetry or transitivity transfer from R_1 to any R_a, if there exists at least one pair $(x_0,y_0)\in X\times X$ such that $x_0R_1y_0$ and $y_0R_1x_0$. Indeed, let for example $x_aR_ay_a$ and $y_aR_ax_a$. Then

$$(ax_a+a^cx_0)R_1(ay_a+a^cy_0) \quad\text{and}\quad (ay_a+a^cy_0)R_1(ax_a+a^cx_0).$$

Hence, $(ax_a+a^cx_0)\sim_1(ay_a+a^cy_0)$, and thus $x_a\sim_a y_a$. Therefore, it is sufficient to require antisymmetry or transitivity only on R_1 if we require reflexivity at the same time.

Definition 3.3.4. A conditional relation $R \sqsubseteq X \times X$ is a *conditional partial order* or a *conditional equivalence relation* on X, if R_1 is a partial order or an equivalence relation on X_1. Let $(X, \leqslant)$ be a conditionally partially ordered set. We say that $Y \in \mathcal{S}(X)$ has a *conditional upper bound, lower bound, supremum, infimum, maximum, minimum, greatest and smallest element* if Y_1 does so with respect to $\leqslant_1$ in X_1. Moreover, Y is said to be *conditionally bounded* if it has a conditional upper and lower bound in X. A conditionally partially ordered set $(X, \leqslant)$ is a *conditional lattice* if there exists a conditional infimum and supremum for every pair x and y in X. If there exists a conditional infimum and supremum for every conditional subset of X which lives on 1, then we say that $(X, \leqslant)$ is a *conditionally complete lattice.*

Note that if $\sim$ is a conditional equivalence relation, then $X_1/{\sim_1}$ is an $\mathcal{A}$-stable subset of X_1, and we denote by $X/{\sim}$ the conditional set which is associated to it. If $(X, \leqslant)$ is a conditionally partially ordered set, then we define $x < y$ whenever $x \leqslant y$ and $ax =_a ay$ implies $a = 0$.

Proposition 3.3.5. *Let $(X, \leqslant)$ be a conditional partial order and $x, y \in X$. Then*

$$\{z \leqslant y\} := \{z \in X : z \leqslant y\}, \qquad \{x \leqslant z \leqslant y\} := \{z \in X : x \leqslant z \leqslant y\},$$
$$\{z < x\} := \{z \in X : z < x\}, \qquad \{x < z < y\} := \{z \in X : x < z < y\},$$

are $\mathcal{A}$-stable subsets of X_1.

Proof. We show the assertion for the last set. Let (a_i) be a partition of unity and $(z_i) \subseteq \{x < z < y\}$. Then $x \leq \sum a_i z_i \leq y$, due to the $\mathcal{A}$-stability of conditional relations. If there exists $a > 0$ such that $a \sum a_i z_i = ax$, then there exists i_0 such that $a \wedge a_{i_0} > 0$. By Lemma 3.1.5, this implies $a \wedge a_{i_0} z_{i_0} =_{a \wedge a_{i_0}} a \wedge a_{i_0} x$ which contradicts the assumptions. The remaining assertions can be shown analogously. ☐

Let $(X, \leqslant)$ be a conditionally partially ordered set. Then $(X_1, \leqslant_1)$ is total if and only if the underlying Boolean algebra $\mathcal{A}$ is trivial or X_1 is a singleton. Indeed, suppose that $x \neq y$ and $x \leqslant y$. Then there exists $a > 0$ such that $ax \neq ay$. Thus, $ax + a^c y$ cannot be compared with $ay + a^c x$.

Definition 3.3.6. A conditionally partially order set $(X, \leqslant)$ is said to be *conditionally total* if for every $x, y \in X$ there exists a partition of unity (a, b, c) such that $ax <_a ay$, $by <_b bx$ and $cx =_c cy$.

By definition, (a, b, c) is uniquely determined for given x and y. Indeed, let $(\tilde{a}, \tilde{b}, \tilde{c})$ be another such triple. Assume $a^c \wedge \tilde{a} > 0$. Then $(a^c \wedge \tilde{a})x <_{a^c \wedge \tilde{a}} (a^c \wedge \tilde{a})y$ and either $(a^c \wedge \tilde{a})y >_{a^c \wedge \tilde{a}} (a^c \wedge \tilde{a})x$ or $(a^c \wedge \tilde{a})x = (a^c \wedge \tilde{a})y$ both of which are contradictory. Hence, $a^c \wedge \tilde{a} = 0$. An analogous argumentation yields $a \wedge \tilde{a}^c = 0$, and thus $a = \tilde{a}$. Similarly, one shows that $b = \tilde{b}$ and $c = \tilde{c}$.

Proposition 3.3.7. *Every conditionally completely ordered set is a conditional lattice.*

Proof. Let $x, y \in X$ and $a, b, c \in \mathcal{A}$ be such that $ax <_a ay$, $by <_b bx$ and $cx =_c cy$. It holds $ax + by + cx \leqslant x$, $ax + by + cx \leqslant y$, $ay + bx + cx \geqslant x$, and $ay + bx + cx \geqslant y$. For any other element $z \in X$ such that $z \leqslant x$ and $z \leqslant y$, it holds $az \leqslant_a ax$, $bz \leqslant_b by$ and $cz \leqslant_c cx$. This implies $z = az + bz + cz \leqslant ax + by + cx$. Thus, $ax + by + cx$ is the greatest lower bound of x and y. By an analogous argument, $ay + bx + cx$ is the least upper bound of x and y. □

The following definition is based on Remarks 3.2.3 and 3.3.2.

Definition 3.3.8. Let S and T be two non-empty sets and $R \subseteq S \times T$ a non-empty binary relation. Then $\mathbf{R} \sqsubseteq \mathbf{S} \times \mathbf{T}$ is a conditional binary relation, called the *canonical extension* of R to $\mathbf{S} \times \mathbf{T}$.

Given $x = \sum a_i x_i \in \mathbf{X}$ and $y = \sum b_j y_j \in \mathbf{Y}$, it holds $x\mathbf{R}y$ if and only if $x_i R y_j$ for every (i, j).

Proposition 3.3.9. *If R is an equivalence relation, a partial order, a total order, a direction, or a lattice, then its canonical extension $\mathbf{R}$ has the respective conditional properties.*

Proof. We prove that the canonical extension of a totally order set $(X, \leq)$ is a conditionally totally ordered set. It holds $\sum a_i x_i \leqslant \sum b_j y_j$ if and only if $x_i \leq y_j$ for all $(i, j) \in I \times J$. Thus, the conditional reflexivity, antisymmetry and transitivity of $\leqslant$ are inherited from the respective properties of $\leq$. Let $\sum a_i x_i, \sum b_j y_j \in \mathbf{X}$, and define

$$a := \vee\{a_i \wedge b_j : x_i < y_j\}, \quad b := \vee\{a_i \wedge b_j : x_i > y_j\}, \quad c := \vee\{a_i \wedge b_j : x_i = y_j\}.$$

Since $\leq$ is a total order on X and $(a_i \wedge b_j) \in \mathcal{K}(1)$, a, b and c are pairwise disjoint and $a \vee b \vee c = 1$. Moreover, $a \sum a_i x_i + b \sum b_j y_j + c \sum a_i x_i \leqslant a \sum b_j y_j + b \sum a_i x_i + c \sum b_j y_j$. The proofs of the remaining assertions follow analogously. □

Examples 3.3.10. The canonical extensions of the usual orders of $\mathbb{N}, \mathbb{Z}$ and $\mathbb{Q}$ are called the *usual* conditional orders on $\mathbf{N}$, $\mathbf{Z}$ and $\mathbf{Q}$, respectively. By Proposition 3.3.9, $(\mathbf{N}, \leqslant), (\mathbf{Z}, \leqslant)$ and $(\mathbf{Q}, \leqslant)$ are conditionally totally ordered sets. ♢

Definition 3.3.11. Let $X = (X_a, \gamma_a)_{a \in \mathcal{A}}$ and $Y = (Y_a, \delta_a)_{a \in \mathcal{A}}$ be two conditional sets. A *conditional function* f from X into Y is a conditional binary relation $G_f \sqsubseteq X \times Y$, where G_{f_a} is the graph of a function $f_a : X_a \to Y_a$ for every $a \in \mathcal{A}$. We denote a conditional function by $f : X \to Y$ and, if there is no risk of confusion, we identify f with f_1. We say that f is *conditionally injective* if $f_a : X_a \to Y_a$ is injective for all $a \in \mathcal{A}$, that it is *conditionally surjective* if for all $y \in Y$ there exists $x \in X$ such that $f_1(x) = y$, and that it is *conditionally bijective* if it is both conditionally injective and surjective. The conditional sets X and Y are said to be *conditionally isomorphic* if there exists a conditional bijection from X into Y.

A conditional function f is a family of functions $(f_a)_{a\in\mathcal{A}}$ where $f_a : X_a \to Y_a$ and such that $f_1(\sum a_i x_i) = \sum a_i f_{a_i}(a_i x_i)$ for all $[a_i, x_i] \subseteq \mathcal{A} \times X$, or, equivalently, such that the diagram

$$\begin{tikzcd} X_1 \arrow[r, "f_1"] \arrow[d, "\gamma_a"'] & Y_1 \arrow[d, "\delta_a"] \\ X_a \arrow[r, "f_a"] & Y_a \end{tikzcd}$$

commutes for every $a \in \mathcal{A}$. Note that if $f = (f_a)_{a\in\mathcal{A}}$ is a conditional function, then $af : aX \to aY$ is a conditional function where $af = (f_b)_{b\in\mathcal{A}_a}$.

Examples 3.3.12. (i) Let (X^i) be a non-empty family of conditional sets on some fixed $\mathcal{A} \in \mathscr{A}$ and $\prod_{i\in I} X^i$ be their conditional product. For every $i \in I$, define $\pi^i : \prod X^i \to X^i$ by $\pi^i := (\pi^i_a)_{a\in\mathcal{A}}$, where $\pi^i_a : \prod_{i\in I} X^i_a \to X^i_a$ is a projection for each $a \in \mathcal{A}$. Since amalgamations in conditional products are evaluated pointwise, it follows that π^i is a conditional function, called the *conditional i-th projection.*

(ii) Let X be a conditional set and $Y \sqsubseteq X$ living on $a \in \mathcal{A}$. The family of functions $f = (f_b)_{b\leq a}$, where each $f_b : Y_b \to X_b$ is an embedding, is a conditional function $f : aY \to aX$, called the *conditional embedding* of Y into X.

(iii) Let $f : X \to Y$ be a conditional function and $Z \sqsubseteq X$ living on a. Let f^Z_b be the restrictions of f_b to Z_b for each $b \leq a$. Then $f^Z = (f^Z_b)_{b\leq a}$ defines a conditional function, due to the $\mathcal{A}_a$-stability of Z. It is called the *conditional restriction* of f to Z. ♢

Proposition 3.3.13. *Let $f : X \to Y$ be a function given by its graph $G \subseteq X \times Y$. Then its canonical extension $\mathbf{G} \sqsubseteq \mathbf{X} \times \mathbf{Y}$ is a conditional function. Moreover, if $f : X \to Y$ is an injection, then its canonical extension is a conditional injection.*

Proof. Since every $\mathbf{G}_a$ is the graph of a function $\mathbf{X}_a \to \mathbf{Y}_a$ and $\mathbf{G}$ is $\mathcal{A}$-stable, $\mathbf{G}$ is a conditional functional relation. Let $f : X \to Y$ be an injection and $\sum a_i x_i, \sum b_j y_j \in \mathbf{X}$ be such that $\sum a_i x_i \sqcap \sum b_j y_j = \mathbf{0}$. Then $x_i \neq y_j$ for all $(i,j) \in I \times J$ with $a_i \wedge b_j > 0$, and thus $f(x_i) \neq f(y_j)$. Since $\mathbf{f} : \mathbf{X} \to \mathbf{Y}$ is a conditional function, it holds $\mathbf{f}(\sum a_i x_i) = \sum a_i f(x_i)$ and $\mathbf{f}(\sum b_j y_j) = \sum b_j f(y_j)$. Hence, $\mathbf{f}(\sum a_i x_i) \sqcap \mathbf{f}(\sum b_j y_j) = \mathbf{0}$. □

Let $(f_a)_{a\in\mathcal{A}}$ be a conditional function from X to Y such that $f_1(x) = y_0$ for all $x \in X$. For every $a \in \mathcal{A}$, this implies $f_a(x) = ay_0$ for all $x \in X_a$. Let $V \in \mathcal{S}(Y)$ be such that $a_* := \vee\{a \in \mathcal{A} : ay_0 \sqsubseteq V\} < 1$. Then $f_1^{-1}(V_1) = \emptyset$, however, $f_{a_*}^{-1}(V_{a_*}) = X_{a_*}$ which motivates the following definition.

Definition 3.3.14. Let $f : X \to Y$ be a conditional function, $U \sqsubseteq X$ and $V \sqsubseteq Y$. The *conditional image* of U is the conditional set $f(U) := \{f_a(x) : x \in U\}$, where a is the condition on which U lives. The *conditional preimage* of V is the conditional set $f^{-1}(V) := \{x \in b_*X : f_{b_*}(x) \in b_*V\}$, where $b_* = \vee\{c \leq b : f_c^{-1}(V_c) \neq \emptyset\}$ and b is the condition on which V lives.

Proposition 3.3.15. *Let $f : X \to Y$ be a conditional function, $[a_i, U^i] \subseteq \mathcal{A} \times \mathcal{P}(X)$ and $[a_i, V^i] \subseteq \mathcal{A} \times \mathcal{P}(Y)$, $(U^j) \subseteq \mathcal{P}(X)$ and $(V^j) \subseteq \mathcal{P}(Y)$, $U \in \mathcal{P}(X)$ and $V \in \mathcal{P}(Y)$, $U^1, U^2 \in \mathcal{P}(X)$ such that $U^1 \sqsubseteq U^2$ and $V^1, V^2 \in \mathcal{P}(Y)$ such that $V^1 \sqsubseteq V^2$. Then it holds*

$$f\Big(\sum a_i U^i\Big) = \sum a_i f(U^i) \qquad f^{-1}\Big(\sum a_i V^i\Big) = \sum a_i f^{-1}(V^i) \tag{3.3.1}$$
$$f(\sqcup U^j) = \sqcup f(U^j) \qquad f^{-1}(\sqcup V^j) = \sqcup f^{-1}(V^j) \tag{3.3.2}$$
$$f(\sqcap U^j) \sqsubseteq \sqcap f(U^j) \qquad f^{-1}(\sqcap V^j) = \sqcap f^{-1}(V^j) \tag{3.3.3}$$
$$f(U)^{\sqsubset} \sqcap f(X) \sqsubseteq f(U^{\sqsubset}) \qquad f^{-1}(V^{\sqsubset}) = f^{-1}(V)^{\sqsubset} \tag{3.3.4}$$
$$f(U^1) \sqsubseteq f(U^2) \qquad f^{-1}(V^1) \sqsubseteq f^{-1}(V^2) \tag{3.3.5}$$
$$U \sqsubseteq f^{-1}(f(U)) \qquad f(f^{-1}(V)) \sqsubseteq V \tag{3.3.6}$$

and an equality holds in (3.3.6) *on the left-hand side if f is conditionally injective, and on the right-hand side if $V \sqsubseteq f(X)$.*

Proof. Since $\mathcal{P}(X)_a \subseteq \mathcal{P}(X)_1$, both diagrams

$$\begin{tikzcd}
\mathcal{P}(X)_1 \arrow[r] \arrow[d] & \mathcal{P}(Y)_1 \arrow[d] & \mathcal{P}(Y)_1 \arrow[r] \arrow[d] & \mathcal{P}(X)_1 \arrow[d] \\
\mathcal{P}(X)_a \arrow[r] & \mathcal{P}(Y)_a & \mathcal{P}(Y)_a \arrow[r] & \mathcal{P}(X)_a
\end{tikzcd}$$

commute for every $a \in \mathcal{A}$, where the horizontal arrows send conditional sets to their conditional image or preimage, respectively. Therefore, $f : \mathcal{P}(X) \to \mathcal{P}(Y)$ and $f^{-1} : \mathcal{P}(Y) \to \mathcal{P}(X)$ define conditional functions which implies (3.3.1) and (3.3.2).

We show (3.3.3). Let a_i be the condition on which U^i lives. Since $f_a(\cap a U^i) \subseteq \cap f_a(a U^i)$ for every $a \in \mathcal{A}$, it holds $\vee\{a \leq \wedge a_i : f_a(\cap a U^i) \neq \emptyset\} \leq \vee\{a \leq \wedge a_i : \cap f_a(a U^i) \neq \emptyset\}$, and thus $f(\sqcap U^i) \sqsubseteq \sqcap f(U^i)$. Let V^i live on a_i and $a_* = \vee\{a \leq \wedge a_i : \cap a V^i \neq \emptyset\}$. Then

$$\vee\{a \leq a_* : f_a^{-1}(\cap a V^i) \neq \emptyset\} = \vee\{a \leq a_* : \cap f_a^{-1}(a V^i) \neq \emptyset\},$$

and this implies $f^{-1}(\sqcap V^i) = \sqcap f^{-1}(V^i)$.

The assertion on the left-hand side of (3.3.4) holds due to the previous steps and the Boolean laws:

$$\begin{aligned}
f(U)^{\sqsubset} \sqcap f(X) &= f(U)^{\sqsubset} \sqcap f(U \sqcup U^{\sqsubset}) \\
&= (f(U)^{\sqsubset} \sqcap f(U)) \sqcup (f(U)^{\sqsubset} \sqcap f(U^{\sqsubset})) \\
&= f(U)^{\sqsubset} \sqcap f(U^{\sqsubset}) \\
&\sqsubseteq f(U^{\sqsubset}).
\end{aligned}$$

It holds $f^{-1}(V^{\sqsubset}) \sqcap f^{-1}(V) = \mathbf{0}$ and $f^{-1}(V^{\sqsubset}) \sqcup f^{-1}(V) = X$, since $f^{-1}(Y) = X$, $f^{-1}(\mathbf{0}) = \mathbf{0}$ and the uniqueness of the conditional complement, and this yields the right-hand side of (3.3.4).

Let U^i live on a_i for $i = 1, 2$. Since $f_{a_1}(U^1) \subseteq f_{a_2}(U^2)$, it follows that $f(U^1) \sqsubseteq f(U^2)$. We proved by the previous steps that f^{-1} is a Boolean homomorphism. In particular, this implies the right-hand side of (3.3.5).

Finally, we prove (3.3.6). Let U live on a. Then $U \subseteq f_a^{-1}(f_a(U))$ and an equality holds if f_a is injective. Thus, $U \sqsubseteq f^{-1}(f(U))$ and an equality holds if f is conditionally injective. As for the right-hand side, let V live on a and $b = \vee\{c \leq a : f_c^{-1}(V_c) \neq \emptyset\}$. Then $f_b(f_b^{-1}(V_b)) \subseteq V_b$ and an equality holds if $V_b \subseteq f_b(X_b)$. Hence, $f(f^{-1}(V)) \sqsubseteq V$ and an equality holds if $V \sqsubseteq f(X)$ since, in particular, $V \subseteq f(X)_a = f_a(X_a)$. □

3.4 Families and cardinality

Definition 3.4.1. Let X and J be two conditional sets on some $\mathcal{A} \in \mathscr{A}$. A *conditional family* $(x_j) = (x_j)_{j \in J}$ is a conditional function $x : J \to X$. The collection of all conditional families $x : J \to X$ defines a conditional set X^J, where X_a^J consists of all conditional restrictions $ax : aJ \to aX$ and the conditioning a sends every conditional family x to its conditional restriction ax for each $a \in \mathcal{A}$. We call X^J the *conditional exponential* of X and J.

Remark 3.4.2. Note that a family $(x_j)_{j \in J} \subseteq X$ is conditional if and only if $\{x_j : j \in J\}$ is in $\mathcal{S}(X)$ since both cases are equivalent to $\sum a_i x_{j_i} = x_{\sum a_i j_i}$. Thus, $\sqcup x_j = \cup x_j$, due to Remark 3.2.15. ♦

Fix a non-degenerate $\mathcal{A} \in \mathscr{A}$.

Lemma 3.4.3. *Let $m = \sum a_i m_i, n = \sum b_j n_j \in \mathbf{N}$ with $m \leqslant n$. Then $\{m \leqslant l \leqslant n\}$ is conditionally isomorphic to $\{1 \leqslant l \leqslant n - m + 1\}$.*[3] *Moreover, for every conditionally bounded $Y \in \mathcal{S}(\mathbf{N})$ there exists a unique $n \in \mathbf{N}$ such that Y is conditionally isomorphic to $\{1 \leqslant l \leqslant n\}$.*

Proof. As for the first assertion, inspection shows that

$$f : \{1 \leqslant l \leqslant n - m + 1\} \to \{m \leqslant l \leqslant n\}, \quad l \mapsto f(l) = l + m - 1,$$

is a conditional bijection. As for the second assertion, suppose that Y is conditionally bounded by $n = \sum b_j n_j$. Let $\mathcal{J}^j$ be the set of subsets of $\{1, \cdots, n_j\} \subseteq \mathbb{N}$. For $J \in \mathcal{J}^j$, define $a_J := \vee\{a \in \mathcal{A} : a \leq b_j \text{ and } aY = J\}$. Then $(a_J)_{J \in \mathcal{J}^j}$ is a partition of b_j. It follows that Y is conditionally isomorphic to $\{1 \leqslant l \leqslant m\}$ where $m = \sum b_j \sum_{J \in \mathcal{J}^j} a_J |J| \in b_j \mathbf{N}$.[4] The uniqueness of m is implied by the first assertion. □

Definition 3.4.4. A conditional set X is said to be *conditionally countable* if there exists a conditional injection $f : X \hookrightarrow \mathbf{N}$. A conditional set X is *conditionally finite* if X is conditionally isomorphic to $\{1 \leqslant k \leqslant n\}$ for some $n \in \mathbf{N}$. In this case we call n the *conditional cardinality* of X. By convention, $\mathbf{0}$ is conditionally finite.

[3] The conditional addition is defined in Section 5.1.
[4] Here, $|J|$ denotes the cardinality of J.

Examples 3.4.5. The conditional sets $\mathbf{Q}$, $\mathbf{N}$ and $\mathbf{Z}$ are conditionally countable, due to Proposition 3.3.13. $\Diamond$

Lemma 3.4.6. *Let $(Y^j)_{j\in J} \subseteq \mathcal{S}(X)$ be a conditionally countable family of conditionally countable subsets. Then $\sqcup_{j\in J} Y^j$ is conditionally countable.*

Proof. Let $f : J \to \mathbf{N}$ be some conditional injection, by assumption there exists one, and define $Z^n := Y^j$ where $n = f(j)$. Then $(Z^n)_{n\in f(J)} = (Y^j)_{j\in J}$ is a conditional family, since the composition of conditional functions is a conditional function. Without loss of generality, assume $f(J) = \mathbf{N}$. Denote by R^n the graph of $Z^n \hookrightarrow \mathbf{N}$ for every $n \in \mathbf{N}$. It follows that $(R^n) \subseteq \mathcal{S}(X \times \mathbf{N})$ is a conditional family. By Remark 3.4.2,

$$R := \sqcup R^n = \cup R^n \sqsubseteq \sqcup Z^n \times \mathbf{N} = \sqcup Y^j \times \mathbf{N}.$$

Then R is the graph of a conditional injection from $\sqcup Z^n$ into $\mathbf{N}$. Indeed, let $x \in \sqcup Z^n$. It follows that $x = \sum a_i x_{n_i}$, where $(a_i) \in \mathcal{K}(1)$ and $x_{n_i} \in Y^{n_i}$ for all i. Hence, there exists a unique $y_{n_i} \in \mathbf{N}$ such that $x_{n_i} R^{n_i} y_{n_i}$, implying that there is a unique $y = \sum a_i y_{n_i} \in \mathbf{N}$ such that xRy. The conditional injectivity of R follows by an analogous argument using the fact that each R^n is the graph of a conditional injection. $\square$

Lemma 3.4.7. *Let X be a conditional set.*

(i) For all $(a_i) \in \mathcal{K}(a)$, $a \in \mathcal{A}$ and every non-empty $(Y^{ij})_{i\in I, j\in J} \subseteq \mathcal{P}(X)$, it holds

$$\sum_{i\in I} a_i \bigsqcup_{j\in J} Y^{ij} = \bigsqcup_{j\in J} \sum_{i\in I} a_i Y^{ij}, \quad \sum_{i\in I} a_i \bigsqcap_{j\in J} Y^{ij} = \bigsqcap_{j\in J} \sum_{i\in I} a_i Y^{ij}.$$

(ii) For every $(a_i)_{i\in I} \in \mathcal{K}(1)$ and $(Y^j)_{j\in J_i} \subseteq \mathcal{P}(X)$ non-empty for each $i \in I$, it holds

$$\sum_{i\in I} a_i \bigsqcup_{j\in J^i} Y^j = \bigsqcup_{j\in \mathbf{J}} Z^j, \quad \sum_{i\in I} a_i \bigsqcap_{j\in J^i} Y^j = \bigsqcap_{j\in \mathbf{J}} Z^j,$$

where $\mathbf{J}$ is generated by $J := \{\sum a_i j_i : j_i \in J_i, i \in I\}$ and $Z^j := \sum b_k \sum a_i Y^{j_{ik}}$ for every $j = \sum b_k \sum a_i j_{ik} \in \mathbf{J}$. In particular,

$$\bigsqcup_{1\leqslant k\leqslant n} Y^k = \sum_{i\in I} a_i \bigsqcup_{j=1}^{n_i} Y^{m_j}, \quad \bigsqcap_{1\leqslant k\leqslant n} Y^k = \sum_{i\in I} a_i \bigsqcap_{j=1}^{n_i} Y^{m_j},$$

where $n = \sum_{i\in I} a_i n_i \in \mathbf{N}$.

(iii) If $(Y^k)_{1\leqslant k\leqslant n}$ is a conditionally finite family of conditionally finite subsets of X, then $\sqcup_{1\leqslant k\leqslant n} Y^k$ is conditionally finite.

Proof. The assertion about the conditional union in (i) follows from Lemma 3.1.5, the assertion about the conditional intersection from the $\mathcal{A}$-stability of the Y^{ij}. The assertion

(ii) is implied by (i), due to Lemma 3.1.5 and the $\mathcal{A}$-stability of conditional families. The representation for conditionally finite conditional intersections and unions is due to

$$\{1 \leqslant m \leqslant n\} = cond\left(\left\{\sum a_i m_i : 1 \leq m_i \leq n_i, m_i \in \mathbb{N}\right\}\right).$$

Finally, the representation for the conditionally finite conditional union in (ii) yields (iii). $\square$

A conditional system $\mathcal{X} \in \mathcal{P}(\mathcal{P}(X))$ is closed under finite conditional intersections or unions if and only if it is already closed under conditionally finite conditional intersections or unions, respectively. We introduce the conventions $\sqcup_{\mathbf{0}} := \mathbf{0}$ and $\sqcap_{\mathbf{0}} := X$.

4 Conditional Topology

In this chapter, we define the conditional version of the basic concepts of topology. This includes topological spaces, bases of a topology, neighborhoods, first and second countability, closure, interior, dense subsets, separability, continuity, filter, nets, convergence and compactness. Topologies which possess a base in $\mathcal{S}(\mathcal{S}(X))$ are closely connected to conditional topologies. This connection allows to compare results from general topology to the corresponding results in conditional topology, and thus can be made fruitful to prove conditional topological results. We introduce the basic concepts of conditional topology and establish the aforementioned connection in Section 4.1. In the remaining sections we prove the conditional version of well-known characterizations of continuity, convergence and compactness. Among the theorems for which we provide a proof are the conditional version of the Ultrafilter Lemma in Section 4.3 and of Tychonoff's Theorem in Section 4.5.

4.1 Topological spaces

Definition 4.1.1. A *conditional topology* on a conditional set X is a family $\mathcal{T}$ in $\mathcal{P}(\mathcal{P}(X))$ satisfying the following properties:

(i) $\mathbf{0}, X \in \mathcal{T}$;

(ii) $\mathcal{T}$ is closed under finite conditional intersections;

(iii) $\mathcal{T}$ is closed under arbitrary conditional unions.

The elements of $\mathcal{T}$ are called *conditionally open sets* and the conditional complement of a conditionally open set is a *conditionally closed set.* A conditional set X endowed with a conditional topology is called a *conditional topological space* and is denoted by $(X, \mathcal{T})$. Given two conditional topologies $\mathcal{T}_1$ and $\mathcal{T}_2$ on X, we say that $\mathcal{T}_1$ is *conditionally weaker* than $\mathcal{T}_2$ (or $\mathcal{T}_2$ is *conditionally stronger* than $\mathcal{T}_1$) if $\mathcal{T}_1 \sqsubseteq \mathcal{T}_2$.

According to Lemma 3.4.7, a conditional topology $\mathcal{T}$ is also closed under conditionally finite intersections. Given $O \in \mathcal{T}$ and $a \in \mathcal{A}$, it holds $aO = aO + a^c\mathbf{0} \in \mathcal{T}$. Therefore, $a\mathcal{T}$ is a conditional topology on aX for every $a \in \mathcal{A}$. The conditional trivial topology $\{aX : a \in \mathcal{A}\}$ and the conditional discrete topology $\mathcal{P}(X)$ are the conditionally weakest and the conditionally strongest conditional topologies on X.

Due to de Morgan's laws, the collection of conditionally closed sets $\mathcal{C} := \{Y^{\sqsubset} : Y \in \mathcal{T}\}$ is a family in $\mathcal{P}(\mathcal{P}(X))$ fulfilling

(i′) $\mathbf{0}, X \in \mathcal{C}$;

(ii′) $\mathcal{C}$ is closed under arbitrary conditional intersection;

(iii′) $\mathcal{C}$ is closed under finite conditional union.

The conditional intersection of any family $(\mathcal{T}_i)$ of conditional topologies on X is itself a conditional topology on X. Indeed, since $X \in \mathcal{T}_i$ for all i, the conditional intersection of the family $(\mathcal{T}_i)$ coincides with their intersection.

Definition 4.1.2. The conditional topology *conditionally generated* by some conditional set $\mathcal{G}$ in $\mathcal{P}(\mathcal{P}(X))$ is defined as

$$\mathcal{T}^{\mathcal{G}} = \bigsqcap \{\mathcal{T} : \mathcal{G} \sqsubseteq \mathcal{T}, \mathcal{T} \text{ conditional topology on } X\}.$$

Recall that $\sqcap_{\mathbf{0}} = X$. Due to the distributivity and associativity of the conditional set operations and Lemma 3.4.7, it holds

$$\mathcal{T}^{\mathcal{G}} := \left\{ \bigsqcup_{i\in I} \bigsqcap_{j\in J_i} O_{ij} : O_{ij} \in \mathcal{G}, J^i \text{ conditionally finite, } I \text{ arbitrary} \right\}.$$

Definition 4.1.3. Let $(X, \mathcal{T})$ be a conditional topological space. A collection of sets $\mathcal{B} \sqsubseteq \mathcal{T}$ is a *conditional topological base* of $\mathcal{T}$ if $\mathcal{B} \in \mathcal{S}(\mathcal{S}(X))$ and for every $O \in \mathcal{T}$ there exist families (a_i) in $\mathcal{A}$ and (O_i) in $\mathcal{B}$ such that $O = \sqcup a_i O_i$.

Observe that for $a = 0$ and any $O \in \mathcal{B}$, it holds $\mathbf{0} = \sqcup 0O$. Observe further that $a\mathcal{B}$ is a conditional topological base of $(aX, a\mathcal{T})$ and $a\mathcal{T}^{\mathcal{B}} = \mathcal{T}^{a\mathcal{B}}$ for every $a \in \mathcal{A}$. Let $\mathcal{B} \sqsubseteq \mathcal{T}$ be such that every conditionally open set is the conditional union of a family in $\mathcal{B}$. Then $\mathcal{B} \cap \mathcal{S}(X)$ is a conditional topological base. Indeed, since there exists a family $(B_i) \subseteq \mathcal{B}$ such that $X = \sqcup B_i$, it holds $\vee b_i = 1$ where B_i lives on b_i for each i. By $\mathcal{A}$-stability and Assumption **(P)**, there exists $B_0 \in \mathcal{B}$ which lives on 1. Let $O \in \mathcal{T}$ and $(B_i) \subseteq \mathcal{B}$ be such that $O = \sqcup B_i$. Then $C_i := b_i B_i + b_i^c B_0 \in \mathcal{B} \cap \mathcal{S}(X)$ for each i, and thus $O = \sqcup b_i C_i$. In particular, $\mathcal{T} \cap \mathcal{S}(X)$ is a conditional topological base of $\mathcal{T}$.

Lemma 4.1.4. *A conditional set $\mathcal{B} \in \mathcal{S}(\mathcal{S}(X))$ is a conditional topological base of $\mathcal{T}^{\mathcal{B}}$ if and only if $\mathcal{B}$ fulfills*

(i) $\sqcup\{O : O \in \mathcal{B}\} = X$;

(ii) for every $O_1, O_2 \in \mathcal{B}$ and $x \in O_1 \sqcap O_2$ there is $O_3 \in \mathcal{B}$ such that $x \in aO_3 \sqsubseteq O_1 \sqcap O_2$, where $O_1 \sqcap O_2$ lives on a.

Proof. Let $\mathcal{B}$ be a conditional topological base of $\mathcal{T}^{\mathcal{B}}$. Then there exist $(a_i) \subseteq \mathcal{A}$ and $(O_i) \subseteq \mathcal{B}$ such that $X = \sqcup a_i O_i \sqsubseteq \sqcup O_i \sqsubseteq \sqcup\{O : O \in \mathcal{B}\}$ which shows *(i)*. To see *(ii)*, let $O_1, O_2 \in \mathcal{B}$. Then there exist $(a_i) \subseteq \mathcal{A}$ and $(O_i) \subseteq \mathcal{B}$ such that $O_1 \sqcap O_2 = \sqcup a_i O_i$. If $x \in O_1 \sqcap O_2$, then there exist $(b_j) \in \mathcal{K}(\vee a_i)$ and $y_j \in a_{i_j} O_{i_j}$ for each j such that $x = \sum b_j y_j$. Hence, $x \in \sum b_j O_{i_j} \sqsubseteq O_1 \sqcap O_2$. By $\mathcal{A}$-stability of $\mathcal{B}$, it holds $\sum b_j O_{i_j} \in \mathcal{B}$.

Conversely, let $\mathcal{B} \in \mathcal{S}(\mathcal{S}(X))$ satisfy *(i)* and *(ii)* and $O \in \mathcal{T}^{\mathcal{B}}$. Then there is a family $(O_{ij})_{j\in J_i, i\in I}$ in $\mathcal{B}$, where I is some arbitrary set and J_i is conditionally finite, such that $O = \sqcup_{i\in I} \sqcap_{j\in J_i} O_{ij}$. If we can show that for every i there exist $a_i \in \mathcal{A}$ and a family $(B_{ik}) \subseteq \mathcal{B}$ such that $\sqcap_{j\in J_i} O_{ij} = a \sqcup B_k$, then the associativity of the conditional union implies $O = \sqcup_{ik} a_i B_{ik}$. Since $\mathcal{B} \in \mathcal{S}(\mathcal{S}(X))$, by Lemma 3.4.7, it suffices to show that for every finite $O_1, O_2, \ldots, O_n \in \mathcal{B}$ there exist $a \in \mathcal{A}$ and $(B_k) \subseteq \mathcal{B}$ such that $\sqcap_{m=1}^{n} O_m = a \sqcup B_k$. To this end, let $O_1, O_2 \in \mathcal{B}$ and define

$$\mathcal{H} = \{O \in \mathcal{B} : aO \sqsubseteq O_1 \sqcap O_2 = aO_1 \cap aO_2\},$$

where the conditional intersection $O_1 \sqcap O_2$ lives on a. By *(ii)*, the collection $\mathcal{H}$ is not empty. Due to the $\mathcal{A}$-stability of $\mathcal{B}$ and of the conditional relation $\sqsubseteq$, it holds $\mathcal{H} \in \mathcal{S}(\mathcal{S}(X))$. Furthermore, *(ii)* implies $O_1 \sqcap O_2 = aO_1 \cap aO_2 = a \sqcup \{O : O \in \mathcal{H}\}$. Finally, induction yields the desired result. $\square$

Throughout this thesis, we assume that a topological base on a set X is a family $\mathfrak{B}$ of subsets of X such that $\emptyset \notin \mathfrak{B}$, $\cup\{O : O \in \mathfrak{B}\} = X$ and for every $O^1, O^2 \in \mathfrak{B}$ and $x \in O^1 \cap O^2$ there exists $O^3 \in \mathfrak{B}$ such that $x \in O^3 \subseteq O^1 \cap O^2$. We write $\mathfrak{T}^{\mathfrak{B}}$ for a topology generated by a family $\mathfrak{B}$ of subsets of some set X. If $\mathcal{B} \in \mathcal{S}(\mathcal{S}(X))$ is a topological base, then the topology it generates contains sets of the form $O_1 \cup O_2$, where $O_1, O_2 \in \mathcal{B}$. In general, $O_1 \cup O_2$ is not an element of $\mathcal{S}(X)$ since it does not include the amalgamation $ax + a^c y$, where $0 < a < 1$ and $x \in O_1$ and $y \in O_2$, if the symmetric difference of O_1 and O_2 is not empty.

Proposition 4.1.5. *Let X be a conditional set.*

(i) If $\mathcal{B}$ is a conditional topological base on X, then $a\mathcal{B}$ is a topological base on aX for every $a \in \mathcal{A}$. Moreover, $\mathcal{T}^{\mathcal{B}} \cap \mathcal{S}(X) = \mathfrak{T}^{\mathcal{B}} \cap \mathcal{S}(X)$.

(ii) If $\mathcal{B} \in \mathcal{S}(\mathcal{S}(X))$ is such that $a\mathcal{B}$ is a topological base on aX for every a, then $\mathcal{B}$ is a conditional topological base on X. Moreover, $\mathfrak{T}^{\mathcal{B}} = \mathfrak{T}^{\mathcal{D}}$ where $\mathcal{D} = \mathcal{T}^{\mathcal{B}} \cap \mathcal{S}(X)$.

Proof. (Step 1) Let $\mathcal{B}$ be a conditional topological base on X. If $(O^i)_{i\in I} \subseteq \mathcal{B}$ is such that $X = \sqcup_{i\in I} O^i$, then it follows by Lemma 3.4.7 that $X = \sqcup_{i\in I} O^i = \cup_{i\in \mathbf{I}} O^i$, since $\mathcal{B} \in \mathcal{S}(\mathcal{S}(X))$. If $O^1, O^2 \in \mathcal{B}$ such that $x \in O^1 \cap O^2$, then $O^1 \cap O^2 = O^1 \sqcap O^2 \in \mathcal{S}(X)$, and by assumption there exists $O^3 \in \mathcal{B}$ such that $x \in O^3 \sqsubseteq O^1 \sqcap O^2$, that is $x \in O^3 \subseteq O^1 \cap O^2$. Hence, $\mathcal{B}$ is a topological base on X, since $\mathcal{S}(X)$ does not contains the empty set. For any $a \in \mathcal{A}$, an analogous argument yields that $a\mathcal{B}$ is a topological base on aX. As for the second assertion, let $O \in \mathfrak{T}^{\mathcal{B}} \cap \mathcal{S}(X)$. Then there exists a family $(O^i) \subseteq \mathcal{B}$ such that $O = \cup O^i$. Since $O \in \mathcal{S}(X)$, it even holds $\cup O^i = \sqcup O^i$. Thus, $O \in \mathcal{T}^{\mathcal{B}}$. Conversely, let $O \in \mathcal{T}^{\mathcal{B}} \cap \mathcal{S}(X)$. Then $O = \sqcup O^i \in \mathcal{S}(X)$ for some family $(O^i) \subseteq \mathcal{B}$. Due to Lemma 3.4.7, it holds $\sqcup_{i\in I} O^i = \cup_{i\in \mathbf{I}} O^i$. Therefore, $O \in \mathfrak{T}^{\mathcal{B}} \cap \mathcal{S}(X)$.

(Step 2) Since $\mathcal{B} \in \mathcal{S}(\mathcal{S}(X))$, it holds $\sqcup\mathcal{B} = \cup\mathcal{B} = X$. Let $x \in O^1 \sqcap O^2 = aO^1 \cap aO^2$, where $O^1 \sqcap O^2$ lives on a. Then $aO^1, aO^2 \in a\mathcal{B}$. By assumption, there exists $O^3 \in \mathcal{B}$ such that $x \in aO^3 \subseteq aO_1 \cap aO^2$. Thus, $x \in aO^3 \sqsubseteq O_1 \sqcap O_2$. By Lemma 4.1.4, $\mathcal{B}$ is

a conditional topological base. As for the second assertion, $\mathcal{B} \subseteq \mathcal{D}$ implies $\mathfrak{T}^{\mathcal{B}} \subseteq \mathfrak{T}^{\mathcal{D}}$. Conversely, for every $O \in \mathcal{D}$ there exists a family $(O^i)_{i\in I} \subseteq \mathcal{B}$ such that $O = \sqcup_{i\in I} O^i$. Hence, $O = \cup_{i\in \mathbf{I}} O^i$, and therefore $O \in \mathfrak{T}^{\mathcal{B}}$ which shows $\mathfrak{T}^{\mathcal{D}} \subseteq \mathfrak{T}^{\mathcal{B}}$. □

Let $(X, \mathcal{T})$ be a conditional topological space and $x \in X$. If $ax \in O$ for some conditionally open set on a, then $x \in aO + a^c X$ which is a conditionally open set on 1. This motivates the following definition.

Definition 4.1.6. Let $(X, \mathcal{T})$ be a conditional topological space and $Y \in \mathcal{S}(X)$. A conditional set $U \sqsubseteq X$ is a *conditional neighborhood* of Y if there exists a conditionally open set O such that $Y \sqsubseteq O \sqsubseteq U$. By $\mathcal{U}(Y)$ we denote the set of all conditional neighborhoods of Y. A *conditional neighborhood base* of an element $x \in X$ is a conditional set $\mathcal{V} \in \mathcal{S}(\mathcal{S}(X))$ such that for every $U \in \mathcal{U}(x)$ there exists $V \in \mathcal{V}$ with $x \in V \sqsubseteq U$.

Note that $a\mathcal{U}(Y) = \mathcal{U}(aY)$. A conditional set $U \in \mathcal{S}(X)$ is a conditional neighborhood of every $x \in U$ if and only if U is conditionally open.

Definition 4.1.7. A conditional topological space $(X, \mathcal{T})$ is *conditionally first countable* if every $x \in X$ has a conditionally countable neighborhood base. It is *conditionally second countable* if $\mathcal{T}$ is conditionally generated by a conditionally countable base. It is *conditionally Hausdorff* if for every pair $x, y \in X$ with $x \sqcap y = \mathbf{0}$ there exists a pair of conditional neighborhoods U_x and U_y such that $U_x \sqcap U_y = \mathbf{0}$. For every $Y \sqsubseteq X$, we define the *conditional interior* of Y by $int(Y) := \sqcup\{O \in \mathcal{T} : O \sqsubseteq Y\}$ and its *conditional closure* by $\mathrm{cl}(Y) := \bigsqcap\{F \in \mathcal{C} : Y \sqsubseteq F\}$. A conditional set $Y \in \mathcal{S}(X)$ is *conditionally dense* in X if $\mathrm{cl}(Y) = X$, and $(X, \mathcal{T})$ is said to be *conditionally separable* if there exists a conditionally countable dense subset of X.

A conditional subset $Y \sqsubseteq X$ is conditionally open if and only if $Y = int(Y)$, and conditionally closed if and only if $cl(Y) = Y$. For every $a \in \mathcal{A}$, it holds $cl(aY) = acl(Y)$ and $int(aY) = aint(Y)$. Furthermore, due to de Morgan's laws,

$$\mathrm{cl}(Y)^{\sqsubset} = int\left(Y^{\sqsubset}\right) \quad \text{and} \quad int(Y)^{\sqsubset} = \mathrm{cl}\left(Y^{\sqsubset}\right). \tag{4.1.1}$$

Proposition 4.1.8. *Let $(X, \mathcal{T})$ be a conditional topological space and $Y \in \mathcal{S}(X)$. Then it holds*

$$int(Y) = \{x \in a_* X : x \in U \sqsubseteq Y \text{ for some } U \in \mathcal{U}(x)\}, \tag{4.1.2}$$
$$cl(Y) = \{x \in X : U \sqcap Y \in \mathcal{S}(X) \text{ for all } U \in \mathcal{U}(x)\}, \tag{4.1.3}$$

where $a_* = \vee\{a \in \mathcal{A} : O \sqsubseteq Y, O \text{ lives on } a\}$.

Proof. Denote the right-hand side of (4.1.2) and (4.1.3) by M and N, respectively. We prove the first identity. By Lemma 3.2.10, a_* is attained. Note that $int(Y)$ lives on a_*. On the one hand, for every $x \in M$ there exists a conditionally open set $O \in \mathcal{U}(x)$ such that $x \in O \sqsubseteq Y$, and thus $x \in int(Y)$. On the other hand, every $x \in int(Y)$ is of the form $x = \sum c_i x_i$ where $x_i \in O^i \sqsubseteq Y$ and $O^i \in \mathcal{T}$ for every i. Hence, $x \in O = \sum c_i O_i \sqsubseteq Y$, and thus $x \in M$. We prove the second identity by the duality $N = int(Y^\sqsubset)^\sqsubset$. By definition, $x \in N$ if $aU \sqcap aY = \mathbf{0}$ implies $a = 0$ for all $U \in \mathcal{U}(x)$. This is equivalent to the statement that $aU \sqsubseteq aY^\sqsubset$ implies $a = 0$ for all $U \in \mathcal{U}(x)$. This is the case if $x \in a int(Y^\sqsubset)$ only if $a = 0$, which is the definition of $x \in int(Y^\sqsubset)^\sqsubset$. □

Let $(X, \mathfrak{T})$ be a topological space. By $\mathfrak{cl}(Y)$ we denote the closure of some $Y \subseteq X$ and by $\mathfrak{int}(Y)$ its interior.

Proposition 4.1.9. *Let X be a conditional set and $\mathcal{B}$ be a conditional topological base for some conditional topology $\mathcal{T}$ on X and $Y \in \mathcal{S}(X)$. Then it holds $cl(Y) = \mathfrak{cl}(Y)$ and $int(Y) = \mathfrak{int}(Y_{a_*})$ where the closure is with respect to $\mathfrak{T}^{\mathcal{B}}$, the interior with respect to $\mathfrak{T}^{\mathcal{B}_{a_*}}$ and a_* as in Proposition 4.1.1.*

Proof. It suffices to argue with a (conditional) base on both sides of both identities. Due to Proposition 4.1.5, it holds that $\mathcal{B}$ and $a_*\mathcal{B}$ are topological bases of $\mathfrak{T}^{\mathcal{B}}$ on X and $\mathfrak{T}^{a_*\mathcal{B}}$ on a_*X, respectively. By Remark 3.2.15, intersection and inclusion coincide with the conditional intersection and inclusion on $\mathcal{S}(X)$. Thus, both assertions follow from the characterizations in Proposition 4.1.1. □

4.2 Continuity

Definition 4.2.1. Let $(X, \mathcal{T})$ and $(X', \mathcal{T}')$ be two conditional topological spaces. A conditional function $f : X \to Y$ is *conditionally continuous* at $x \in X$ if $f^{-1}(U)$ is a conditional neighborhood of x for every conditional neighborhood U of $f(x)$. A conditional function is said to be *conditionally continuous* on X if it is conditionally continuous at every $x \in X$. Two conditional sets X and X' are said to be *conditionally homeomorphic* if there exists a conditional bijection $f : X \to X'$ such that f and f^{-1} are both conditionally continuous.

Recall that if $f : X \to X'$ is a conditional function, then so is $f_a : aX \to aX'$. Now observe that if f is conditionally continuous at $x \in X$, then so is $f_a : aX \to aX'$ at $ax \in aX$ with respect to the topologies $a\mathcal{T}$ and $a\mathcal{T}'$.

Proposition 4.2.2. *Let $f : X \to X'$ be a conditional function between two conditional topological spaces. Then the following statements are equivalent:*

(i) f is conditionally continuous;

(ii) $f^{-1}(O)$ is conditionally open in X for every conditionally open set O in X';

(iii) $f^{-1}(F)$ is conditionally closed in X for every conditionally closed set F in X';

(iv) $f^{-1}(int(Z)) \sqsubseteq int(f^{-1}(Z))$ for every $Z \sqsubseteq X'$;

(v) $f(cl(Z)) \sqsubseteq cl(f(Z))$ for every $Z \sqsubseteq X$.

Proof. Due to (3.3.5), f is conditionally continuous at $x \in X$ if and only if $f^{-1}(U)$ is an element of a conditional neighborhood base of x for every U in a conditional neighborhood base of $f(x)$. Thus, *(ii)* implies *(i)*. To show that *(i)* implies *(ii)*, recall that $f^{-1}(O) = \{x \in a_* X : f(x) \in a_* O\}$ where $a_* = \vee\{a \in \mathcal{A} : f_a^{-1}(O_a) \neq \emptyset\}$. For any $x \in f^{-1}(O)$, $a_* O$ is a conditional neighborhood of $f(x) \in \mathcal{S}(a_* f(x))$. Hence, $f^{-1}(O)$ is a conditional neighborhood of x. Therefore, $f^{-1}(O)$ is conditionally open since it is the conditional neighborhood of all of its elements. From (3.3.4) follows the equivalence of *(ii)* and *(iii)*. Next, we show the equivalence of *(ii)* and *(iv)*. Assume *(ii)*. For every $Z \sqsubseteq X'$, it holds $\mathrm{i}nt(Z) \sqsubseteq Z$. Hence, $f^{-1}(\mathrm{i}nt(Z)) \sqsubseteq f^{-1}(Z)$, due to (3.3.5). By assumption, $f^{-1}(\mathrm{i}nt(Z)) = \mathrm{i}nt(f^{-1}(\mathrm{i}nt(Z))) \sqsubseteq \mathrm{i}nt(f^{-1}(Z))$. To show that *(iv)* implies *(ii)*, let O be conditionally open in X'. Since $\mathrm{i}nt(O) = O$, it holds

$$\mathrm{i}nt(f^{-1}(O)) \sqsubseteq f^{-1}(O) = f^{-1}(\mathrm{i}nt(O)) \sqsubseteq \mathrm{i}nt(f^{-1}(O)).$$

Finally, we show that *(iii)* and *(v)* are equivalent. Assume *(iii)*. Then $f^{-1}(\mathrm{c}l(f(Z)))$ is conditionally closed. Since $f(Z) \sqsubseteq \mathrm{c}l(f(Z))$, it follows that $Z \sqsubseteq f^{-1}(\mathrm{c}l(f(Z)))$, by (3.3.5). Hence, $\mathrm{c}l(Z) \sqsubseteq f^{-1}(\mathrm{c}l(f(Z)))$, and by (3.3.5) and (3.3.6), it holds

$$f(\mathrm{c}l(Z)) \sqsubseteq f(f^{-1}(\mathrm{c}l(f(Z)))) \sqsubseteq \mathrm{c}l(f(Z)).$$

Conversely, for every conditionally closed subset F of X', from (3.3.5) and (3.3.6) it follows that

$$\mathrm{c}l(f^{-1}(F)) \sqsubseteq f^{-1}(f(\mathrm{c}l(f^{-1}(F)))) \sqsubseteq f^{-1}(\mathrm{c}l(f(f^{-1}(F)))) \sqsubseteq f^{-1}(\mathrm{c}l(F)) = f^{-1}(F).$$

Since $f^{-1}(F) \sqsubseteq \mathrm{c}l(f^{-1}(F))$, it follows that $f^{-1}(F)$ coincides with its conditional closure. □

Proposition 4.2.3. *Let $(X, \mathcal{T})$ and $(X', \mathcal{T}')$ be two conditional topological spaces, and $\mathcal{B}$ and $\mathcal{B}'$ be their conditional topological bases, respectively. A conditional function $f : X \to X'$ is conditionally continuous if and only if f is continuous with respect to the topologies $\mathfrak{T}^{\mathcal{B}}$ and $\mathfrak{T}^{\mathcal{B}'}$.*

Proof. By Lemma 4.1.5, the sets $\mathcal{B}$ and $\mathcal{B}'$ are topological bases on X and X', respectively. Assume that f is continuous. Let $O \in \mathcal{B}'$, and $f^{-1}(O)$ lives on $a \in \mathcal{A}$. Then

$$a f_a^{-1}(O) + a^c X = f_1^{-1}(aO + a^c Y) \in \mathfrak{T}^{\mathcal{B}} \cap \mathcal{S}(X) = \mathcal{T}^{\mathcal{B}} \cap \mathcal{S}(X),$$

by Lemma 4.1.5. Thus, $f^{-1}(O) = a f^{-1}(aO + a^c Y) \in a\mathcal{T}^{\mathcal{B}}$, and therefore $f^{-1}(O)$ is conditionally open in X. Conversely, assume that f is conditionally continuous and let $O \in \mathcal{B}'$. If $f^{-1}(O) = \emptyset$, we are done. Suppose that $f^{-1}(O) \neq \emptyset$. Then $f^{-1}(O)$ in an element of $\mathcal{T}^{\mathcal{B}} \cap \mathcal{S}(X)$. Thus, $f^{-1}(O) \in \mathfrak{T}^{\mathcal{B}}$, due to Lemma 4.1.5. □

Definition 4.2.4. Let X be a conditional set, $(X_i, \mathcal{T}_i)$ be a family of conditional topological spaces and (f_i) be a family of conditional functions $f_i : X \to X_i$. The conditional topology $\mathcal{T}$ on X conditionally generated by $cond(\mathcal{G})$ where

$$\mathcal{G} := \{f_i^{-1}(O_i) : O_i \in \mathcal{T}_i \text{ for some } i \in I\},$$

is called the *conditional initial topology* on X for the family (f_i).

By construction, the conditional initial topology $\mathcal{T}$ on X for the family (f_i) is the conditionally coarsest topology for which every f_i is conditionally continuous. The conditional initial topology can be equally generated by considering the $\mathcal{A}$-stable hull of the conditional preimages of a conditional base of each of the $\mathcal{T}_i$.

Proposition 4.2.5. *Let X be a conditional set, $(X_i, \mathcal{T}_i, \mathcal{B}_i)_{i\in I}$ be a conditional (finite) family of conditional topological spaces with bases B_i, where $X_i \in \mathcal{S}(X)$ for each $i \in I$, and $(f_i)_{i\in I}$ be a conditional (finite) family of conditional functions $f_i : X \to X_i$. Let $\mathfrak{T}$ be the initial topology on X for the family (f_i) and $\mathcal{T}$ be the conditional initial topology on X for the family (f_i). Then it holds $\mathcal{T} = \mathcal{T}^{\mathfrak{B}}$ and $\mathfrak{T} = \mathfrak{T}^{\mathfrak{B}} = \mathfrak{T}^{\mathcal{B}} = \mathfrak{T}^{\mathcal{T}\cap\mathcal{S}(X)}$ where $\mathcal{B} = cond(\mathfrak{B})$ and $\mathfrak{B} = \{f_i^{-1}(O_i) : O_i \in \mathcal{B}_i \text{ for some } i \in I\} \cap \mathcal{S}(X)$.*

Proof. Due to Lemma 4.1.5, it suffices to show that $\mathfrak{B} = \mathcal{B}$ in both cases. If $(f_i)_{i\in I}$ and $(\mathcal{T}_i)_{i\in I}$ are conditional families, then $\sum a_j f_{i_j}^{-1}(O_{i_j}) = f^{-1}_{\sum a_j i_j}(O_{\sum a_j i_j}) \in \mathfrak{B}$, showing that $\mathfrak{B}$ is $\mathcal{A}$-stable. Next, let I be finite. Without loss of generality, $|I| = 2$. We need to show that $\sum a_j f_{i_j}^{-1}(O_{i_j}) \in \mathfrak{T}$ for every $[a_j, f_{i_j}^{-1}(O_{i_j})] \subseteq \mathcal{A} \times \mathfrak{B}$, where $i_j \in \{1, 2\}$. To this end, it suffices to show that $af_1^{-1}(O_1) + a^c f_2^{-1}(O_2) \in \mathfrak{T}$ for every $O_i \in \mathcal{B}_i$ such that $f_i^{-1}(O_i) \in \mathfrak{B}$ for each $i = 1, 2$. It holds

$$af_1^{-1}(O_1) + a^c f_2^{-1}(O_2) = f_1^{-1}(aO_1 + a^c X_1) \cap f_2^{-1}(a^c O_2 + aX_2). \tag{4.2.1}$$

Indeed, let $x = ax_1 + a^c x_2$ be an element of the left-hand side of (4.2.1). Then

$$f_1(x) = af_1(x_1) + a^c f_1(x_2) \in aO_1 + a^c X_1,$$
$$f_2(x) = a^c f_2(x_2) + af_2(x_1) \in a^c O_2 + aX_2.$$

Hence, x is an element of the right-hand side of (4.2.1). Since the intersection is finite, the last term is an element of $\mathfrak{T}$. Conversely, let $x \in f_1^{-1}(aO_1 + a^c X_1) \cap f_2^{-1}(a^c O_2 + aX_2)$. It follows that $f(x) = y_i$ where $y_1 \in aO_1 + a^c X_1$ and $y_2 \in a^c O_2 + aX_2$. Hence, ax is in $af_1^{-1}(aO_1 + a^c X_1)$ and $a^c x$ in $a^c f_2^{-1}(a^c O_2 + aX_2)$, and thus x is an element of the left-hand side of (4.2.1). $\square$

Examples 4.2.6. (i) *The conditional relative topology.* Let $(X, \mathcal{T})$ be a conditional topological space, $Y \sqsubseteq X$ lives on a, and $f : aY \to aX$ the conditional embedding. The conditional initial topology of $a\mathcal{T}$ for f is called the conditional relative topology with respect to Y. Note that a conditionally open set in Y is the conditional intersection of Y with a conditionally open set in X.

(ii) *The conditional least upper bound of a family of conditional topologies.* Let $(\mathcal{T}_i)_{i\in I}$ be a family of conditional topologies on $X = X_i$ and $f_i = \mathrm{id}$ for all $i \in I$. The conditional initial topology corresponds to the conditionally weakest conditional topology on X which conditionally includes every $\mathcal{T}_i$.

(iii) *The conditional product topology.* Let $(X_i, \mathcal{T}_i)_{i\in I}$ be a family of conditional topological spaces, where X_i is a conditional set on some fixed $\mathcal{A}$ for all $i \in I$. The conditional product topology on $X = \prod_{i\in I} X_i$ is the conditional initial topology for the family of conditional projections $(\pi_i)_{i\in I}$. ◊

4.3 Filters and nets

Throughout this section we assume that X is a conditional set on some non-degenerate $\mathcal{A} \in \mathscr{A}$.

Definition 4.3.1. If $(J, \leqslant)$ is a conditionally directed set on $\mathcal{A}$, then we call the conditional family $(x_j)_{j\in J} \sqsubseteq X$ a *conditional net.* If $(K, \leqslant)$ is another conditionally directed set, a conditional net $(y_\beta)_{\beta\in K}$ is called a *conditional subnet* of $(x_\alpha)_{\alpha\in J}$ if there exists a conditional function $\phi : K \to J$ such that $x_{\phi(\beta)} = y_\beta$, and for any $\alpha_0 \in J$ there exists $\beta_0 \in K$ such that $\beta \geqslant \beta_0$ implies $\phi(\beta) \geqslant \alpha_0$. If $(J, \leqslant) = (\mathbf{N}, \leqslant)$, we say that $(x_\alpha) = (x_n)$ is a *conditional sequence.*

Recall that $\mathcal{P}(\mathcal{P}(X))$ is the conditional set of conditional systems of conditional subsets of X and $\mathcal{S}(\mathcal{S}(X)) \sqsubseteq \mathcal{P}(\mathcal{P}(X))$ which represents the conditional systems of conditional subsets of X for which each element lives on 1.

Definition 4.3.2. A *conditional filter* on X is a non-empty family $\mathcal{F} \in \mathcal{S}(\mathcal{S}(X))$ satisfying

(i) $Z \in \mathcal{F}$ whenever $Y \sqsubseteq Z$ for some $Y \in \mathcal{F}$ and $Z \in \mathcal{S}(X)$;

(ii) $Y \sqcap Z \in \mathcal{F}$ for all $Y, Z \in \mathcal{F}$.

A conditional filter $\mathcal{F}$ is called *conditionally free* if $\sqcap\{Y : Y \in \mathcal{F}\} = \mathbf{0}$, otherwise it is called *conditionally principal.*

Observe that $X \in \mathcal{F}$ and $\mathbf{0} \notin \mathcal{F}$. Furthermore, $a\mathcal{F}$ is a conditional filter on aX for any $a > 0$, due to Property (ii). Every conditional filter is a filter base since, by (i) and (ii), the conditional intersection and inclusion within $\mathcal{S}(X)$ coincide with the intersection and inclusion, see Remark 3.2.15.

Examples 4.3.3. (i) The *conditional principal filter* $\mathcal{F}$ is a conditional filter of the form $\mathcal{F} = \{Z \in \mathcal{S}(X) : Y \sqsubseteq Z\}$ for some $Y \in \mathcal{S}(X)$. In particular, the *conditional trivial filter* $\{X\}$ is a conditional principal filter.

(ii) Let $(X, \mathcal{T})$ be a conditional topological space. Then the conditional set $\mathcal{U}(Y)$ of all conditional neighborhoods of Y is a conditional filter on X. ◊

Remark 4.3.4. Let $\mathcal{F} \in \mathcal{P}(\mathcal{P}(X))$ satisfy

(i) $\mathbf{0} \notin \mathcal{F}$;

(ii) $\mathcal{F}$ is closed under finite conditional intersection;

(iii) $Z \in \mathcal{F}$ whenever $Y \sqsubseteq Z$ for some $Y \in \mathcal{F}$ and $Z \in \mathcal{P}(X)$.

Then there exists a minimal condition $m_{\mathcal{F}} > 0$ such that $m_{\mathcal{F}}\mathcal{F} = \mathcal{F} \cap \mathcal{S}(m_{\mathcal{F}}X)$ is a conditional filter on $m_{\mathcal{F}}X$. Indeed, let $\mathcal{F}$ be given by $\{Y^i : i \in I\}$ and suppose, for the sake of contradiction, that $\wedge a_i = 0$. By de Morgan's laws, it holds $\vee a_i^c = 1$. Let $(b_j)_{j\in J} \in \mathcal{K}(1)$ be such that, according to **(P)**, $b_j \leq a_{i_j}^c$ for some $i_j \in I$ and all $j \in J$. Then $\sum_{j\in J} b_j Y^{i_j} = \mathbf{0}$. Since $\mathcal{F}$ is a conditional set, it follows that $\sum_{j\in J} b_j Y^{i_j} \in \mathcal{F}$ which contradicts $\mathbf{0} \notin \mathcal{F}$. ♦

Proposition 4.3.5. *Let $x = (x_\alpha)_{\alpha\in J} \subseteq X$ be a conditional net and $\mathcal{F}$ be a conditional filter on X. Then $\mathcal{B}^x := \{(x_\beta)_{\beta\geqslant\alpha} : \alpha \in J\}$ is a conditional filter base for a conditional filter $\mathcal{F}^x$. Furthermore, $\{(y, Y) : Y \in \mathcal{F}, y \in Y\}$ is a conditionally directed set, where $(y^1, Y^1) \geqslant (y^2, Y^2)$ whenever $Y^1 \sqsubseteq Y^2$. Setting $x^{\mathcal{F}}_{(y,Y)} := y$ defines a conditional net $x^{\mathcal{F}}$. Finally, $x \sqsubseteq x^{\mathcal{F}^x}$ and $\mathcal{F}^{x^{\mathcal{F}}} = \mathcal{F}$.*

Proof. Since $\{\beta : \beta \geqslant \alpha\}$ is $\mathcal{A}$-stable, $(x_\beta)_{\beta\geqslant\alpha} \in \mathcal{S}(X)$ for every α. Further, x being a conditional net implies $\sum a_i (x_\beta)_{\beta\geqslant\alpha_i} = (x_\beta)_{\beta\geqslant\sum a_i\alpha_i} \in \mathcal{B}^x$ showing that $\mathcal{B}^x \in \mathcal{S}(\mathcal{S}(X))$. The conditional intersection property holds since J is a conditional direction.

The fact that $\{(y, Y) : Y \in \mathcal{F}, y \in Y\}$ is due to $\sqsubseteq$ being a conditional partial order. Hence, $x^{\mathcal{F}}$ is a net. As for the last assertion, let $Y \in \mathcal{F}$. It follows that

$$\{x^{\mathcal{F}}_{(\tilde{y},\tilde{Y})} : (\tilde{y}, \tilde{Y}) \geqslant (y, Y)\} = Y \in \mathcal{B}^{x^{\mathcal{F}}}.$$

Hence, $\mathcal{F} \sqsubseteq \mathcal{F}^{x^{\mathcal{F}}}$. Analogously, it follows that $\mathcal{B}^{x^{\mathcal{F}}} \sqsubseteq \mathcal{F}$, and thus $\mathcal{F} = \mathcal{F}^{x^{\mathcal{F}}}$. Let now $\alpha \in J$. Then, $x^{\mathcal{F}^x}_{(x_\alpha,(x_\beta)_{\beta\geqslant\alpha})} = x_\alpha$ showing that $x \sqsubseteq x^{\mathcal{F}^x}$. □

Definition 4.3.6. A non-empty family $\mathcal{B} \in \mathcal{S}(\mathcal{S}(X))$ is called a *conditional filter base* if for every $Y^1, Y^2 \in \mathcal{B}$ there exists $Y^3 \in \mathcal{B}$ such that $Y^3 \sqsubseteq Y^1 \sqcap Y^2$. For every filter base $\mathcal{B}$, the family $\mathcal{F}^{\mathcal{B}} := \{Z \sqsubseteq X : Y \sqsubseteq Z \text{ for some } Y \in \mathcal{B}\}$ is a conditional filter, referred to as the conditional filter *conditionally generated* by $\mathcal{B}$.

Given a filter base $\mathfrak{B}$ on X, we write $\mathfrak{F}^{\mathfrak{B}}$ for the filter generated by $\mathfrak{B}$.

Proposition 4.3.7. *Let X be a conditional set.*

(i) If $\mathcal{B}$ is a conditional filter base on X, then $a\mathcal{B}$ is a filter base on aX for every $a \in \mathcal{A}$. Moreover, it holds $\mathcal{F}^{\mathcal{B}} = \mathfrak{F}^{\mathcal{B}} \cap \mathcal{S}(X)$.

(ii) If $\mathcal{B} \in \mathcal{S}(\mathcal{S}(X))$ is filter base on X, then $\mathcal{B}$ is a conditional filter base on X. Moreover, it holds $\mathfrak{F}^{\mathcal{B}} = \mathfrak{F}^{\mathcal{F}^{\mathcal{B}}}$.

Proof. For a conditional filter base $\mathcal{B} \subseteq \mathcal{S}(X)$ on X, it holds $\emptyset \notin \mathcal{B}$ and for any $Y^1, Y^2 \in \mathcal{B}$ there exists $Y^3 \in \mathcal{B}$ such that $Y^3 \sqsubseteq Y^1 \sqcap Y^2$. According to Remark 3.2.15, it follows that $\mathcal{B}$ is a filter base on X. Analogously, one shows that $a\mathcal{B}$ is a filter base on aX. Let $Y \in \mathcal{F}^{\mathcal{B}}$, then $Y \in \mathcal{S}(X)$. Furthermore, there exists $Z \in \mathcal{B}$ with $Z \sqsubseteq Y$. By Remark 3.2.15, $Z \subseteq Y$ and so $Y \in \mathfrak{F}^{\mathcal{B}}$. The converse inclusion follows analogously.

As for the second assertion, let $\mathcal{B} \in \mathcal{S}(\mathcal{S}(X))$ be a filter base. According to Remark 3.2.15, $\mathcal{B}$ is also a conditional filter base on X. From $\mathcal{B} \subseteq \mathcal{F}^{\mathcal{B}}$ it follows that $\mathfrak{F}^{\mathcal{B}} \subseteq \mathfrak{F}^{\mathcal{F}^{\mathcal{B}}}$. For $Y \in \mathfrak{F}^{\mathcal{F}^{\mathcal{B}}}$ there exists $Z \in \mathcal{F}^{\mathcal{B}}$ such that $Z \subseteq Y$. Since $\mathcal{B}$ is a conditional base, there exists $W \in \mathcal{B}$ such that $W \subseteq Z \subseteq Y$, by Remark 3.2.15. Hence, $Y \in \mathfrak{F}^{\mathcal{B}}$, and thus $\mathfrak{F}^{\mathcal{B}} = \mathfrak{F}^{\mathcal{F}^{\mathcal{B}}}$. □

Definition 4.3.8. Given two conditional filters $\mathcal{F}$ and $\mathcal{F}'$ on a conditional set X, we say that $\mathcal{F}'$ is *conditionally finer* than $\mathcal{F}$ (or that $\mathcal{F}$ is *conditionally coarser* than $\mathcal{F}'$) if $\mathcal{F} \sqsubseteq \mathcal{F}'$. Conditional filters on X are conditionally partially ordered by the conditional relation "$\mathcal{F}'$ is conditionally finer than $\mathcal{F}$". A conditional filter $\mathcal{F}$ on X which is conditionally maximal is a *conditional ultrafilter.*

We prove the conditional version of the Ultrafilter lemma.

Theorem 4.3.9. *For every conditional filter $\mathcal{F}$ there exists a conditional ultrafilter $\mathcal{U}$ which is conditionally finer than $\mathcal{F}$. Furthermore, if $\mathfrak{U}$ is an ultrafilter of $\mathfrak{F}^{\mathcal{F}}$, then $\mathcal{U}^{\mathfrak{U}} = cond(\mathfrak{U} \cap \mathcal{S}(X))$ is a conditional ultrafilter of $\mathcal{F}$. Finally, if $\mathcal{U}$ is a conditionally principal ultrafilter, then $\mathcal{U} = \{Y \sqsubseteq X : x \in Y\}$ for a unique $x \in X$.*

Proof. (Step 1) Let $\mathscr{F}$ be the set of conditional filters which are conditionally finer than $\mathcal{F}$. Let us show that $\mathscr{F}$ has a conditional maximal element. Notice that $\mathscr{F}$ is $\mathcal{A}$-stable. Let $(\mathcal{F}^j) \subseteq \mathscr{F}$ be a chain and define $\mathcal{U} := \sqcup \mathcal{F}^j \in \mathcal{U} \in \mathcal{S}(\mathcal{S}(X))$. Let now $Z \sqsubseteq X$ be such that $Y \sqsubseteq Z$ for some $Y = \sum a_i Y^{j_i} \in \mathcal{U}$ where $(a_i) \in \mathcal{K}(1)$ and $Y^{j_i} \in \mathcal{F}^{j_i}$ for all i. It follows that $a_i Y^{j_i} \sqsubseteq a_i Z$ for every i, and therefore $a_i Z \in a_i \mathcal{F}^{j_i}$, that is $a_i Z + a_i^c X \in \mathcal{U}$ for every i. By $\mathcal{A}$-stability of $\mathcal{U}$, it follows that $Z = \sum a_i \left(a_i Z + a_i^c X\right) \in \mathcal{U}$. The conditional intersection property holds by a similar argumentation, and thus $\mathcal{U} \in \mathscr{F}$. The existence of a conditional maximal element follows by Zorn's Lemma.

(Step 2) Let $\mathfrak{U}$ be an ultrafilter of $\mathfrak{F}^{\mathcal{F}}$. Since $\mathcal{F} \subseteq \mathfrak{F}^{\mathcal{F}} \subseteq \mathfrak{U}$ and $\mathcal{F} \in \mathcal{S}(\mathcal{S}(X))$, it follows that $\mathcal{F} \subseteq \mathcal{U}^{\mathfrak{U}}$. The maximality of $\mathcal{U}^{\mathfrak{U}}$ follows from the conditional maximality of $\mathfrak{U}$.

(Step 3) Let $\mathcal{U}$ be a conditional principal ultrafilter and $x \in \sqcap\{Y \sqsubseteq X : Y \in \mathcal{U}\} \neq \mathbf{0}$. Since $\mathcal{U}_x = \{Y \sqsubseteq X : x \in Y\}$ is a conditional filter, it holds $\mathcal{U} \sqsubseteq \mathcal{U}_x$. Hence, $\mathcal{U} = \mathcal{U}_x$. Finally, $\{x\} = \sqcap\{Y \sqsubseteq X : Y \in \mathcal{U}_x\} = \sqcap\{Y \sqsubseteq X : Y \in \mathcal{U}\}$ yields the uniqueness of x. □

Proposition 4.3.10. *Let $\mathcal{U}$ be a conditional filter on X. Then the following assertions are equivalent:*

(i) $\mathcal{U}$ is a conditional ultrafilter;

(ii) if $Y^1 \sqcup Y^2 \in \mathcal{U}$ for some $Y^1, Y^2 \sqsubseteq X$, then $aY^1 + a^c Y^2 \in \mathcal{U}$, where either $a = a_1$ or $a = a_2^c$ whereby a_i is the condition on which Y^i lives, $i = 1, 2$;

(iii) *for every $Y \sqsubseteq X$, it holds $aY + a^c Y^{\sqsubset} \in \mathcal{U}$, where either $a = a_1$ or $a = a_2^c$ whereby a_1 and a_2 are the conditions on which Y and $Y^{\sqsubset}$ live, respectively;*

(iv) *for every $Y \in \mathcal{S}(X)$ such that $Y \sqcap U \in \mathcal{S}(X)$ for every $U \in \mathcal{U}$, it holds $Y \in \mathcal{U}$.*

Proof. *(Step 1)* We show that (i) implies (ii). Following Remark 3.2.15, let $b_1 := a_1 \wedge a_2^c$, $b_2 := a_1 \wedge a_2$ and $b_3 := a_2 \wedge a_1^c$. Since $Y^1 \sqcup Y^2 = b_1 Y^1 + (b_2 Y^1 \sqcup b_2 Y^2) + b_3 Y^2$, it follows that $b_2 Y^1 \sqcup b_2 Y^2 \in b_2 \mathcal{U}$. If $b_2 Y^1 \in b_2 \mathcal{U}$, then $a = b_1 \vee b_2 = a_1$, and we are done. If $b_2 Y^1 \notin b_2 \mathcal{U}$, it follows that $\mathcal{F} = \{Z \in \mathcal{S}(b_2 X) : Z \sqcup b_2 Y^1 \in b_2 \mathcal{U}\}$ is a conditional filter on $b_2 X$ and contains $b_2 Y^2$ which lives on b_2. Furthermore, $b_2 \mathcal{U} \sqsubseteq \mathcal{F}$, and therefore, from $\mathcal{U}$ being a conditional ultrafilter it follows that $b_2 Y^2 \in b_2 \mathcal{U}$. Thus, $a^c = b_2 \vee b_3 = a_2$ does the job.

(Step 2) To show that (ii) implies (iii), it suffices to set $Y^1 := Y$ and $Y^2 := Y^{\sqsubset}$ since $Y \sqcup Y^{\sqsubset} = X \in \mathcal{U}$.

(Step 3) We show that (iii) implies (i). Let $\mathcal{V}$ be a conditional filter conditionally finer than $\mathcal{U}$. For every $Y \in \mathcal{V}$, it holds $Y \in \mathcal{S}(X)$. By assumption, either $Y \in \mathcal{U}$ or $aY + a^c Y^{\sqsubset} \in \mathcal{U}$, where $Y^{\sqsubset}$ lives on $a^c > 0$. The latter case implies $a^c Y^{\sqsubset} \in a^c \mathcal{U} \subseteq a^c \mathcal{V}$, and therefore, since $Y \in \mathcal{V}$, both $a^c Y$ and $a^c Y^{\sqsubset}$ are in $a^c \mathcal{V}$. Yet, a^c being strictly positive, this implies that $a^c \mathcal{V}$ is a conditional filter on $a^c X$. However, then $a^c Y \sqcap a^c Y^{\sqsubset} = \mathbf{0}$ which is impossible. Hence, $a^c = 0$ and therefore $Y \in \mathcal{U}$, showing that $\mathcal{V} \sqsubseteq \mathcal{U}$.

(Step 4) We finally show that (i) is equivalent to (iv). Assume (i) and let $Y \in \mathcal{S}(X)$ be such that $Y \sqcap Y \in \mathcal{S}(X)$ for every $U \in \mathcal{U}$. By assumption, $\mathcal{B} := \{Y \sqcap U : U \in \mathcal{U}\}$ is in $\mathcal{S}(\mathcal{S}(X))$. Inspection shows that $\mathcal{B}$ is a conditional filter base on X. It holds further that $\mathcal{U} \sqsubseteq \mathcal{F}^{\mathcal{B}}$. Hence, $\mathcal{U} = \mathcal{F}^{\mathcal{B}}$, and thus $Y \in \mathcal{U}$. Conversely, let $\mathcal{V}$ be a conditional ultrafilter of $\mathcal{U}$ and let $Y \in \mathcal{V}$. From $\mathcal{U} \sqsubseteq \mathcal{V}$ it follows that $Y \sqcap U \in \mathcal{S}(X)$ for every $U \in \mathcal{U}$, and therefore, $Y \in \mathcal{U}$ showing that $\mathcal{V} \sqsubseteq \mathcal{U}$. □

Proposition 4.3.11. *Let $f : X \to Y$ be a conditional function. For every conditional filter $\mathcal{F}$ on X, the family $f(\mathcal{F}) := \{f(U) : U \in \mathcal{F}\}$ is a conditional filter base on Y. Moreover, if $\mathcal{U}$ is a conditional ultrafilter on X then $f(\mathcal{U})$ generates a conditional ultrafilter on Y.*

Proof. By (3.3.1), since $\mathcal{F} \in \mathcal{S}(\mathcal{S}(X))$, it holds $f(\mathcal{F}) \in \mathcal{S}(\mathcal{S}(Y))$. Let $V^1 = f(U^1)$, $V^2 = f(U^2)$ and $V^3 = f(U^1 \sqcap U^2)$ where $U^1, U^2 \in \mathcal{F}$. Since $\mathcal{F}$ is a conditional filter and by (3.3.3), it follows $V^1 \sqcap V^2 \sqsupseteq V^3 \in f(\mathcal{F})$. Thus, $f(\mathcal{F})$ is a conditional filter base.

As for the second claim, suppose that $\mathcal{U}$ is a conditional ultrafilter on X and denote by $\mathcal{V}$ the conditional filter generated by $f(\mathcal{U})$. Let $V \sqsubseteq Y$ live on a_1, and $V^{\sqsubset}$ live on a_2. By (3.3.2) and (3.3.4), it holds

$$f^{-1}(V) \sqcup f^{-1}(V)^{\sqsubset} = f^{-1}(V) \sqcup f^{-1}(V^{\sqsubset}) = f^{-1}(V \sqcup V^{\sqsubset}) = f^{-1}(Y) = X \in \mathcal{U}.$$

By Proposition 4.3.10, it follows that $U := af^{-1}(V) + a^c f^{-1}(V)^{\sqsubset} \in \mathcal{U}$ where either $a = a_1$ or $a = a_2^c$. Due to (3.3.2), (3.3.4) and (3.3.6), it holds

$$f(\mathcal{U}) \ni f(U) = af\left(f^{-1}(V)\right) + a^c f\left(f^{-1}(V^{\sqsubset})\right) \sqsubseteq aV + a^c V^{\sqsubset}.$$

Hence, for every $V \sqsubseteq Y$, it holds $aV + a^c V^{\sqsubset} \in \mathcal{V}$, where either V lives on a or $V^{\sqsubset}$ lives on a^c. Thus, Proposition 4.3.10 implies that $\mathcal{V}$ is a conditional ultrafilter. □

4.4 Convergence

Definition 4.4.1. Let X be a conditional topological space, $\mathcal{F}$ be a conditional filter on X, and $(x_\alpha) \subseteq X$ be a conditional net. An element $x \in X$ is

(i) a *conditional limit point* of $\mathcal{F}$, if $U \in \mathcal{F}$ for every conditional neighborhood U of x;

(ii) a *conditional cluster point* of $\mathcal{F}$, if $x \in \mathrm{cl}(Y)$ for every $Y \in \mathcal{F}$;

(iii) a *conditional limit point* of (x_α), if for every conditional neighborhood U of x there exists α_0 such that $(x_\alpha)_{\alpha \geqslant \alpha_0} \sqsubseteq U$;

(iv) a *conditional cluster point* of (x_α), if for every conditional neighborhood U of x and every α there exists $\beta \geqslant \alpha$ such that $x_\beta \in U$.

We define $\mathrm{Lim}\, \mathcal{F} = \sqcap\{\mathrm{cl}(Y) : Y \in \mathcal{F}\}$ and $\mathrm{Lim}(x_\beta) = \sqcap\{cl((x_\beta)_{\beta \geqslant \alpha}) : \alpha\}$.

If $\mathrm{Lim}\, \mathcal{F}$ lives on a, then it is the collection of all conditional cluster points of $a\mathcal{F}$. Note that if x is a conditional limit or cluster point of a conditional filter $\mathcal{F}$ or net (x_α) with respect to the conditional topology $\mathcal{T}$ on X, then ax is a conditional limit or cluster point of the conditional filter $a\mathcal{F}$ or the conditional net (ax_α) with respect to the topology $a\mathcal{T}$ on aX. We indicate by $\mathcal{F} \to x$ and by $x_\alpha \to x$ that x is a conditional limit point of $\mathcal{F}$ and (x_α), respectively. For a topology $\mathfrak{T}$ on X and a filter $\mathfrak{F}$ on X, we denote by $\mathfrak{F} \to x$ the convergence of $\mathfrak{F}$ to x and by $\mathfrak{Lim}\mathfrak{F}$ the set of cluster points of $\mathfrak{F}$ with respect to $\mathfrak{T}$. Analogously, for a net $(x_\alpha) \subseteq X$ we denote by $x_\alpha \xrightarrow{\mathfrak{T}} x$ the convergence of (x_α) to x and by $\mathfrak{Lim}(x_\alpha)$ the set of cluster points of (x_α).

Proposition 4.4.2. *Let $(X, \mathcal{T})$ be a conditional topological space and $\mathcal{F}$ be a conditional filter on X. The following assertions are equivalent:*

(i) $x \in \mathrm{Lim}\, \mathcal{F}$;

(ii) There exists a conditional filter $\mathcal{G} \sqsupseteq \mathcal{F}$ such that $\mathcal{G} \to x$.

Proof. To show that *(i)* implies *(ii)*, let $x \in \mathrm{Lim}\, \mathcal{F}$. Then $\{V \sqcap U : V \in \mathcal{U}(x) \text{ and } U \in \mathcal{F}\}$ is a conditional filter base of a conditional filter $\mathcal{G} \sqsupseteq \mathcal{F}$. By construction, $\mathcal{G} \to x$. To show that *(ii)* implies *(i)*, let $\mathcal{G}$ be a conditional filter conditionally finer than $\mathcal{F}$ and $\mathcal{G} \to x$. Hence, $V \sqcap Y \in \mathcal{S}(X)$ for every $V \in \mathcal{U}(x)$ and every $Y \in \mathcal{F}$ since both are elements of $\mathcal{G}$. Thus, the characterization (4.1.3) of the conditional closure implies that $x \in \mathrm{cl}(Y)$, showing that $x \in \mathrm{Lim}\, \mathcal{F}$. □

Corollary 4.4.3. *Let $(X,\mathcal{T})$ be a conditional topological space and $(x_\alpha) \sqsubseteq X$ be a conditional net. The following assertions are equivalent:*

(i) $x \in \mathrm{Lim}(x_\alpha)$.

(ii) *There exists a conditional subnet (x_β) of (x_α) such that $x_\beta \to x$.*

Proposition 4.4.4. *Let $(X,\mathcal{T})$ be a conditional topological space with conditional base $\mathcal{B}$ and $\mathcal{F}$ be a conditional filter on X. Then $x \in \mathrm{Lim}\,\mathcal{F}$ if and only if $x \in \mathfrak{Lim}\mathfrak{F}^{\mathcal{F}}$, where $\mathfrak{T} = \mathfrak{T}^{\mathcal{B}}$.*

Proof. Let $x \in \mathfrak{Lim}\mathfrak{F}^{\mathcal{F}}$. By Proposition 4.3.7, from $Y \in \mathcal{F}$ it follows that $Y \in \mathfrak{F}^{\mathcal{F}}$. Hence, $x \in \mathfrak{cl}(Y)$. Due to Proposition 4.1.9, it follows that $x \in cl(Y)$. Thus, $x \in \mathrm{Lim}\,\mathcal{F}$. Conversely, let $x \in \mathrm{Lim}\,\mathcal{F}$, that is $x \in cl(Y)$ for every $Y \in \mathcal{F}$. By Proposition 4.3.7, $\mathcal{F}$ is a filter base of $\mathfrak{F}^{\mathcal{F}}$, and thus it suffices to show that $x \in \mathfrak{cl}(Y)$ for every $Y \in \mathcal{F}$. Since $Y \in \mathcal{S}(X)$, Proposition 4.1.9 yields $x \in cl(Y) = \mathfrak{cl}(Y)$. □

Corollary 4.4.5. *Let $(X,\mathcal{T})$ be a conditional topological space with conditional base $\mathcal{B}$ and $(x_\alpha) \sqsubseteq X$ be a conditional net. Then $x \in \mathrm{Lim}(x_\alpha)$ if and only if $x \in \mathfrak{Lim}(x_\alpha)$, where $\mathfrak{T} = \mathfrak{T}^{\mathcal{B}}$.*

Proposition 4.4.6. *Let $(x_\alpha) \sqsubseteq X$ be a conditional net and $\mathcal{B} \in \mathcal{S}(\mathcal{S}(X))$ be such that $a\mathfrak{B}$ is a topological base for every $a \in \mathcal{A}$. Then $x_\alpha \xrightarrow{\mathfrak{T}^{\mathcal{B}}} x$ if and only if $x_\alpha \xrightarrow{\mathcal{T}} x$. In particular, $Y \in \mathcal{S}(X)$ is conditionally closed if and only if $x_\alpha \to x \in Y$ for every conditionally converging net $(x_\alpha) \sqsubseteq Y$.*

Proof. The equivalence between convergence and conditional convergence follows from Propositions 4.1.5 and 4.3.7. This together with Proposition 4.1.9 yields the characterization of conditional closedness. □

Corollary 4.4.7. *Let $\mathcal{G} \in \mathcal{S}(\mathcal{S}(X))$ be a conditional filter, $\mathcal{B} \in \mathcal{S}(\mathcal{S}(X))$ be such that $a\mathcal{B}$ is a topological base for every $a \in \mathcal{A}$ and $\mathfrak{F}$ be the filter generated by $\mathcal{G}$. Then it holds $\mathfrak{F} \to x$ if and only if $\mathcal{F}^{\mathfrak{F}} \to x$. In particular, $Y \in \mathcal{S}(X)$ is conditionally closed if and only if $\mathrm{Lim}\,\mathcal{F} \sqsubseteq Y$ for every conditional filter $\mathcal{F}$ on Y.*

Remark 4.4.8. Note that by the definition of a conditional filter and a conditional net, if $(X,\mathcal{T})$ is a conditional Hausdorff space, then the conditional limit of any conditionally converging net and any conditionally converging filter is unique. Moreover, if X is conditionally separable, then there exists a conditionally countable $Y \in \mathcal{S}(X)$ such that $cl(Y) = X$. Hence, there exists for every $x \in X$ a conditionally countable neighborhood base. Thus, it suffices to argue with conditional sequences in statements about conditional convergence. ♦

Proposition 4.4.9. *Let X and Y be two conditional topological spaces and $f : X \to Y$ be a conditional function. Then the following assertions are equivalent:*

(i) *f is conditionally continuous at x;*

(ii) *$f(x_\alpha) \to f(x)$ for every conditional net $x_\alpha \to x$;*

(iii) *$f(\mathcal{F}) \to f(x)$ for every conditional filter $\mathcal{F} \to x$.*

Proof. By Proposition 4.2.3, conditional continuity is equivalent to continuity. Hence, the claim follows from [Bou98, Chapter 7.4, Proposition 9], Proposition 4.4.6 and Corollary 4.4.7. □

Corollary 4.4.10. *The composition of conditionally continuous functions is conditionally continuous.*

4.5 Compactness

Definition 4.5.1. Let X be a conditional topological space. A *conditional open covering* of X is a conditional family $(O^j) \subseteq \mathcal{T}$ such that $X \sqsubseteq \sqcup O^j$. A conditional family $(Y^j)_{j\in J} \subseteq \mathcal{P}(X)$ has the *conditional finite intersection property* if $\sqcap_{j\in N} Y^j \neq \mathbf{0}$ for every conditionally finite $N \subseteq J$.

We introduce the concept of conditional compactness.

Definition 4.5.2. A conditional topological space X is *conditionally compact* if for every conditional open covering $(O^j)_{j\in J}$ it holds

$$X \sqsubseteq \bigsqcup_{1\leqslant k\leqslant n} O^{j_k} \tag{4.5.1}$$

for some conditionally finite subfamily $(O^{j_k})_{1\leqslant j_k\leqslant n}$. In a conditional topological space $(X, \mathcal{T})$, we say that Y is a conditionally compact subset of X if Y is a conditionally compact with respect to the conditional relative topology on Y.

Due to Lemma 3.4.7, (4.5.1) is equivalent to

$$X \sqsubseteq \sum a_i \bigsqcup_{j\in J^i} O^j$$

for some partition $(a_i) \subseteq \mathcal{A}$ and $J^i \subseteq J$ finite for each i.

Proposition 4.5.3. *Let $(X, \mathcal{T})$ be a conditional topological space. Then the following statements are equivalent:*

(i) *X is conditionally compact;*

(ii) *every conditional filter on X has a conditional cluster point;*

(iii) *every conditional ultrafilter on X has a conditional limit point;*

(iv) every conditional family (F^j) of conditionally closed sets with the conditionally finite intersection property is such that $\sqcap F^j \neq \mathbf{0}$.

Proof. The equivalence between (ii) and (iii) follows from Proposition 4.4.2. Suppose now that X is conditionally compact and let (F^j) be a conditional family of conditionally closed sets with the conditional finite intersection property. Suppose that $\sqcap F^j = \mathbf{0}$. By de Morgan's laws, it follows that $\sqcup O^j = X$ where $O^j = (F^j)^{\sqsubset} \in \mathcal{T}$. Hence, the conditional family (O^j) is a conditional open covering of X. Therefore, there exists $n \in \mathbf{N}$ such that $X = \sqcup_{1 \leqslant j \leqslant n} O^j$, from which it follows that $\mathbf{0} = \sqcap_{1 \leqslant j \leqslant n} F^j$, contradicting the hypothesis. Conversely, suppose that (iv) holds and let (O^j) be a conditional open covering of X. It follows that $\sqcap F^j = \mathbf{0}$ for the conditional family of conditionally closed sets $F^j = (O^j)^{\sqsubset}$. Hence, there exists a conditional finite family N such that $\sqcap_{j \in N} F^j = \mathbf{0}$, otherwise there is contradiction with (iv). Taking conditional complements yields (i). Suppose that (ii) holds, and let (F^j) be a conditional family of conditionally closed sets with the conditional finite intersection property. Let $\mathcal{G} = \{\sqcap_{j \in N} F^j : N \text{ conditionally finite}\}$. Since (F^j) is a conditional family of $\mathcal{A}$-stable sets, it follows that $\mathcal{G} \in \mathcal{P}(\mathcal{P}(X))$. Furthermore, by the conditional finite intersection property, it follows that $\mathbf{0} \notin \mathcal{B}$. Let $a > 0$ be the minimal condition of $\mathcal{G}$, see Remark 4.3.4. Let now $Y^1 = \sqcap_{j \in N} F^j, Y^2 = \sqcap_{j \in M} F^j \in a\mathcal{G}$. By Lemma 3.4.7, $N \sqcup M$ is conditionally finite. Then it holds $Y^1 \sqcap Y^2 = \sqcap_{j \in N \sqcup M} F^j \in a\mathcal{G}$. Hence, $a\mathcal{G}$ is a conditional filter base of a conditional filter $\mathcal{F}^{a\mathcal{G}}$ on aX. Since $\mathcal{F} = \mathcal{F}^{a\mathcal{G}} + a^c\{X\}$ is a filter on X, by assumption, there exists $x \in \operatorname{Lim} \mathcal{F}$, and therefore $ax \in \operatorname{Lim} \mathcal{F}^{a\mathcal{G}}$. However, $\sqcap_j F^j \sqsupseteq \sqcap\{cl(Y) : Y \in \mathcal{F}^{a\mathcal{G}}\} = \operatorname{Lim} \mathcal{F}^{a\mathcal{B}} \ni ax$ showing that (iv) holds. Conversely, suppose that (iv) holds, and let $\mathcal{F}$ be a conditional filter on X. Note that $\mathrm{cl}(\sqcap Y^i) \sqsubseteq \sqcap \mathrm{cl}(Y^i)$. Hence, it follows, due to Lemma 3.4.7, that the family $\{\mathrm{cl}(Y) : Y \in \mathcal{F}\}$ is a conditional family of conditionally closed sets fulfilling the conditional finite intersection property. Thus, $\operatorname{Lim} \mathcal{F} = \sqcap\{\mathrm{cl}(Y) : Y \in \mathcal{F}\} \neq \mathbf{0}$. Let therefore $a > 0$ be the condition on which $\operatorname{Lim} \mathcal{F}$ lives. Assume, for the sake of contradiction, that $a < 1$. It follows that $a^c\mathcal{F}$ is a conditional filter on a^cX. By the same argumentation as above, it follows that $\{\mathrm{cl}(Y) : Y \in a^c\mathcal{F}\}$ is a family of conditionally closed sets in a^cX fulfilling the conditional finite intersection property. Hence, $\operatorname{Lim} a^c\mathcal{F} = \sqcap\{cl(Y) : Y \in a^c\mathcal{F}\} \neq \mathbf{0}$. Let $\operatorname{Lim} a^c\mathcal{F}$ live on b. By $\mathcal{A}$-stability $\operatorname{Lim} \mathcal{F}$ lives on $a \vee b$ which contradicts the definition of a. Thus, $\operatorname{Lim} \mathcal{F} \in \mathcal{S}(X)$. ☐

Proposition 4.5.4. *Let $(X, \mathcal{T})$ and $(X', \mathcal{T}')$ be two conditionally compact topological space.*

(i) Every conditionally finite subset of X is conditionally compact.

(ii) Conditionally finite unions of conditionally compact subsets are conditionally compact.

(iii) A conditionally closed subset of a conditionally compact subset is conditionally compact.

(iv) If X is a conditionally Hausdorff space, then every conditionally compact subset Y is conditionally closed.

(v) Every conditionally continuous function $f : X \to X'$ carries conditionally compact subsets to conditionally compact subsets.

Proof. (i) Let $(x_k)_{1\leqslant k\leqslant n}$ be a conditionally finite subset of X and (O^i) be a conditional open covering of $(x_k)_{1\leqslant k\leqslant n}$. Since (O^i) is a conditional family, $x_k \in O^{i_k}$ for some i_k. Thus, $(x_k)_{1\leqslant k\leqslant n} \sqsubseteq \sqcup_{1\leqslant k\leqslant n} O^{i_k}$.

(ii) Follows from Lemma 3.4.7.

(iii) Let $Z \sqsubseteq Y$ be a conditionally closed subset of the conditionally compact set Z and (O^i) be a conditional open covering of Z. Then the conditional family $(\tilde{O}^j)$ generated by (O^i) and $Z^\sqsubset$ is a conditional open covering of Y, and there exists a conditionally finite subfamily $(\tilde{O}^{j_k})$ which conditionally covers Y. By the representation in Lemma 3.4.7, one can extract from $(\tilde{O}^{j_k})$ those elements which are on some a_i equal to $Z^\sqsubset$ where $k = \sum a_i k_i$. The resulting conditional family is a conditional open covering of Z.

(iv) Let $x \in Y^\sqsubset$. We will show that there exists a conditionally open set O_x such that $x \in O_x \sqsubseteq Y^\sqsubset$. For every $y \in Y$, the conditional Hausdorff property implies the existence of conditionally disjoint open sets $U(y)$ and $V(y)$ such that $x \in U(y)$ and $y \in V(y)$. Then the collection $\{V(y) : y \in Y\}$ is a conditional open covering of Y since Y is $\mathcal{A}$-stable. By assumption, there exists a conditionally finite subfamily $(V(y_k))_{1\leqslant k\leqslant n}$ which conditionally covers Y. Moreover, it holds $x \in O_x := \sqcap_{1\leqslant k\leqslant n} U(y_k)$ which is conditionally open since it is the conditional intersection of a conditionally finite family of conditional open sets, due to Lemma 3.4.7. Let $z \in O_x$, then $z \in U(y_k)$ for all $1 \leqslant k \leqslant n$. Since $U(y_k) \sqcap V(y_k) = \mathbf{0}$ and $Y = \sqcup_{1\leqslant k\leqslant n} V(y_k)$, it follows that $z \sqcap Y = \mathbf{0}$. Thus, $O_x \sqsubseteq Y^\sqsubset$. Then $Y^\sqsubset = \sqcup_{x\in Y^\sqsubset} O_x$ which is conditionally open and so Y is conditionally closed.

(v) Follows from (3.3.2) and (3.3.6). □

We prove the conditional version of Tychonoff's theorem.

Theorem 4.5.5. *Let $(X^i, \mathcal{T}^i)_{i\in I}$ be a family of conditional topological spaces and $X = \prod_{i\in I} X^i$ be endowed with the conditional product topology. Then X is conditionally compact if and only if X^i is conditionally compact for every $i \in I$.*

Proof. By definition of the conditional product topology, all the conditional projections $\pi_i : X \to X^i$ are conditionally continuous. Therefore, if X is conditionally compact, so is $X^i = \pi_i(X)$ for every $i \in I$. Conversely, let $\mathcal{U}$ be a conditional ultrafilter on X. By Proposition 4.3.11, it follows that $\pi_i(\mathcal{U})$ is a conditional ultrafilter on X^i for each $i \in I$. Due to the conditional compactness of X^i, it holds that $\pi_i(\mathcal{U}) \to x_i$. For every conditionally open neighborhood $O^i \in \mathcal{T}^i$ of x_i, which implies that $O^i \in \mathcal{S}(X^i)$, it follows that $O^i \sqcap \pi_i(U) \in \mathcal{S}(X^i)$ for every $i \in I$. Using (3.3.3) and (3.3.6) and the conditional injectivity of π_i, it follows that $\pi_i^{-1}(O^i) \sqcap U \in \mathcal{S}(X)$ for every conditionally open neighborhood $O^i \in \mathcal{T}^i$ of x_i and every $U \in \mathcal{U}$ for all $i \in I$. By Proposition 4.3.10, it follows that $\pi^{-1}(O^i) \in \mathcal{U}$ for every conditionally open neighborhood $O^i \in \mathcal{T}^i$ of x_i. Since $\mathcal{U}$ is $\mathcal{A}$-stable, it follows that any conditional neighborhood of x is an element of $\mathcal{U}$, showing that $\mathcal{U} \to x$. □

5 Conditional Real Numbers

In this chapter, we develop the basics of real analysis on conditional sets. The main part consists of the construction of the conditional real numbers. Our construction is based on the construction of real numbers by rational Cauchy sequences, due to Cantor. To this end, we introduce conditionally ordered fields in Section 5.1. In Section 5.2, we construct the conditional real numbers $\mathbf{R}$ and show that $\mathbf{R}$ is a conditionally ordered field such that every conditionally bounded subset of $\mathbf{R}$ has a conditional infimum and supremum. We show that the conditional Euclidean topology of $\mathbf{R}$ is conditionally complete and separable. In Section 5.3, we discuss the connection of $\mathbf{R}$ to the conditional set $\mathbf{L^0}$ associated to the set of measurable functions on a σ-finite measure space, which establishes the link to the existing literature on L^0-theory. In Section 5.4, we prove the conditional version of Debreu's Gap Lemma. We use $\mathbf{R}$ to introduce conditional metric spaces in Section 5.5. Among the theorems we prove in this chapter are the conditional version of the Bolzano-Weierstraß, the Heine-Borel and the Borel-Lebesgue Theorem.

5.1 Ordered fields

Recall that we identify X with X_1.

Definition 5.1.1. Let X be a conditional set. We call $(X,+)$ a *conditional group* if $+ : X \times X \to X$ is a conditional function and $(X,+)$ is a group. A conditional subset $Y \sqsubseteq X$ is a *conditional subgroup* if $Y \in \mathcal{S}(X)$ and $(Y,+)$ is a subgroup of X. We call $(X,+,\cdot)$ a *conditional ring* if $+ : X \times X \to X$ and $\cdot : X \times X \to X$ are conditional functions and $(X,+,\cdot)$ is a ring. A conditional subset $Y \sqsubseteq X$ is a *conditional subring* if $Y \in \mathcal{S}(X)$ and $(Y,+,\cdot)$ is a subring of X.

Observe that if $(X,+,\cdot)$ is a conditional ring, where X is a conditional set on some $\mathcal{A} \in \mathscr{A}$, then also $(aX, +^{aX\times aX}, \cdot^{aX\times aX})$ is a conditional ring on $\mathcal{A}_a$ for every $a \in \mathcal{A}$. We denote "$x \cdot y$" by "xy". If there is no risk of confusion, we use the same notation for group addition and amalgamations, as well the multiplication in a ring and the conditioning action. The same holds for the notation 0 and 1 for the neutral elements of the addition and the multiplication, respectively, and the distinguished elements 0 and 1 of some Boolean algebra $\mathcal{A}$. The following properties are immediate consequences of the previous definition.

(i) For all $[a_i, x_i], [b_j, y_j] \subseteq \mathcal{A} \times X$,

$$\sum a_i x_i + \sum b_j y_j = \sum (a_i \wedge b_j)(x_i + y_j);$$

(ii) $a(x+y) = ax + ay$ for all $x, y \in X$ and $a \in \mathcal{A}$;

(iii) for all $[a_i, x_i], [b_j, y_j], [c_l, z_l] \subseteq \mathcal{A} \times X$,

$$\sum a_i x_i \Big(\sum b_j y_j + \sum c_l z_l \Big) = \sum (a_i \wedge b_j \wedge c_l)(x_i y_j + x_i z_l);$$

(iv) $a(x(y+z)) = axay + axaz$ for all $x, y, z \in X$ and $a \in \mathcal{A}$.

Examples 5.1.2. (i) Let $(X, +, \cdot)$ be a conditional ring and I be a conditional set on the same Boolean algebra $\mathcal{A}$. Then $(X^I, +, \cdot)$ where

$$(x+y)(i) := x(i) + y(i) \quad \text{and} \quad (x \cdot y)(i) = x(i) \cdot y(i)$$

is a conditional ring, due to Lemma 3.1.5.

(ii) Let $(S, +, \cdot)$ be a ring. The canonical extension of $+$ and $\cdot$ is defined by

$$\sum a_i x_i + \sum b_j y_j := \sum (a_i \wedge b_j)(x_i + y_j), \quad \sum a_i x_i \cdot \sum b_j y_j := \sum (a_i \wedge b_j)(x_i \cdot y_j),$$

respectively, and in this case, $(\mathbf{S}, +, \cdot)$ is a conditional ring. As such the conditional rings of the conditional integers $(\mathbf{Z}, +, \cdot)$ and the conditional rational numbers $(\mathbf{Q}, +, \cdot)$ are defined. $\diamondsuit$

If a conditional field $(X, +, .)$ is defined as a conditional set X and two conditional functions $+$ and $\cdot$ such that $(X, +, \cdot)$ is a field, then $(X, +, \cdot)$ is a field only if $\mathcal{A}$ is the trivial algebra, since otherwise there exists an inverse of $a1 + a^c 0$ for some $0 < a < 1$, which is absurd.

Definition 5.1.3. A conditional ring $(X, +, .)$ is a *conditional field* if for every x in $X^* := \mathbf{0}^{\sqsubset}$ there exists $y \in X^*$ such that $xy = yx = 1$. We call y the *conditional inverse* of x.

By the same argument as in the classical case, if a conditional inverse exists, then it is unique.

Example 5.1.4. The conditional ring $\mathbf{Q}$ is a conditional field. Indeed, the conditional inverse of some $q = \sum a_i q_i \in \mathbf{Q}^*$ ($q_i \in \mathbb{Q}^*$ for every $i \in I$) is given by $1/q := \sum a_i 1/q_i$. $\diamondsuit$

Definition 5.1.5. Let X and K be two conditional sets on the same Boolean algebra $\mathcal{A}$. We call X a *conditional K-module* if K is a conditional ring, X is a K-module and the scalar operation $\cdot : K \times X \to X$ is a conditional function. If K is a conditional field, X is called a *conditional K-vector space*.

Example 5.1.6. Let K be a conditional ring and I be a conditional set on the same Boolean algebra $\mathcal{A}$. Then K^I is a conditional K-module where

$$(f+g)(i) := f(i) + g(i) \quad \text{and} \quad (\lambda f)(i) := \lambda(i) f(i).$$

In particular, if K is a conditional field and $n \in \mathbf{N}$ with $n = \sum a_i n_i$, then

$$K^n := \sum a_i \prod_{1 \leq k_i \leq n_i} K^{k_i}$$

is a conditional K-vector space. ◊

Definition 5.1.7. Let $(R, +, \cdot, \leqslant)$ be a conditional ring and a conditionally totally ordered set. We say that R is a *conditionally ordered ring* if

(i) $x < y$ implies $x + z < y + z$ for all $z \in R$;

(ii) $x, y > 0$ implies $xy > 0$.

In particular, if R is a conditional field, we say that R is a *conditionally ordered field.*

Examples 5.1.8. (i) The conditional rational numbers $(\mathbf{Q}, +, \cdot, \leqslant)$ with their conditional usual order are a conditionally ordered field.

(ii) The conditional sequences $(\mathbf{Q}^{\mathbf{N}}, +, \cdot, \leqslant)$, where $(q_n) \leqslant (p_n)$ if $q_n \leqslant p_n$ for every $n \in \mathbf{N}$, form a conditionally ordered ring. ◊

Definition 5.1.9. Let $(K, +, \cdot, \leqslant)$ be a conditionally ordered field and

$$K_+ := \{x \in K : x \geqslant 0\}.$$

The *conditional absolute value* on K is defined by

$$|\cdot| : K \to K_+, \quad x \mapsto \max\{x, 0\}.$$

The following properties are immediate consequences of the definition.

(i) $|\cdot|$ is a conditional function;

(ii) $|x| = 0$ if and only if $x = 0$;

(iii) $|-x| = |x|$ for all $x \in K$;

(iv) $x = |x|$, if $x \in K_+$;

(v) $||x| - |y|| \leqslant |x + y| \leqslant |x| + |y|$ for every $x, y \in K$;

(vi) and $|xy| = |x|\,|y|$ for all $x, y \in K$.

The conditional inequality $|x + y| \leqslant |x| + |y|$ is called the *conditional triangular inequality.*

Example 5.1.10. Let $(\mathbf{Q}, +, \cdot, \leqslant)$ be the conditionally ordered field of conditional rational numbers. Then $|\cdot| : \mathbf{Q} \to \mathbf{Q}_+$ is given by

$$q = \sum a_i q_i \longmapsto |q| = \sum a_i \, |q_i| \, .$$ ◊

5.2 The real line

Fix a non-degenerate $\mathcal{A} \in \mathscr{A}$.

Definition 5.2.1. Let (x_n) be a conditional sequence in some conditionally partially ordered set $(X, \leqslant)$. We say that (x_n) is *conditionally increasing (decreasing)* if $m \leqslant n$ implies $x_m \leqslant x_n$ $(x_m \geqslant x_n)$. The sequence (x_n) is called *conditionally strictly increasing (decreasing)* if $m < n$ implies $x_m < x_n$ $(x_m > x_n)$. We say that (x_n) is *conditionally (strictly) monotone* if there exists $a \in \mathcal{A}$ such that $(a x_n)$ is conditionally (strictly) increasing and $(a^c x_n)$ is conditionally (strictly) decreasing.

Let $(\mathbf{Q}, +, \cdot, \leqslant)$ be the conditionally ordered field of conditional rational numbers and $(\mathbf{Q}^{\mathbf{N}}, +, \cdot, \leqslant)$ be the conditionally ordered ring of conditional sequences in $\mathbf{Q}$. For every $\varepsilon \in \mathbf{Q}$ with $\varepsilon > 0$ there exists $n \in \mathbf{N}$ such that $0 < 1/n < \varepsilon$. Indeed, for $\varepsilon = \sum a_i q_i$ choose some $n_i \in \mathbb{N}$ for each q_i such that $0 < 1/n_i < q_i$. Then $0 < 1/n < \varepsilon$ where $n = \sum a_i n_i$.

Proposition 5.2.2. *For $n \in \mathbf{N}$ and $q \in \mathbf{Q}$, define*

$$B_{\frac{1}{n}}(q) = \left\{ p \in \mathbf{Q} : |q - p| < \frac{1}{n} \right\}.$$

Then $\mathcal{B} := \{B_{1/n}(q) : q \in \mathbf{Q}, n \in \mathbf{N}\}$ is a conditional topological base of a conditionally Hausdorff topology on $\mathbf{Q}$, and $\mathbf{Q}$ endowed with $\mathcal{T}^{\mathcal{B}}$ is conditionally separable.

Proof. Let (a_i) be a partition of unity and $(p_i) \subseteq \mathcal{B}_{1/n}(q)$ for some $n \in \mathbf{N}$ and $q \in \mathbf{Q}$. By $\mathcal{A}$-stability, $|\sum a_i p_i - q| < 1/n$. Let further $(B_{1/n_i}(q_i)) \subseteq \mathcal{B}$. Then

$$\sum a_i B_{\frac{1}{n_i}}(q_i) = B_{\frac{1}{\sum a_i n_i}}\Big(\sum a_i q_i\Big) \in \mathcal{B}.$$

Let $B_{1/n}(q), B_{1/m}(p) \in \mathcal{B}$ and $x \in B_{1/n}(q) \sqcap B_{1/m}(p)$. Let $q = \sum a_i q_i$, $n = \sum b_j n_j$, $p = \sum c_k p_k$, $m = \sum d_l m_l$ and $x = \sum e_t x_t$. Without loss of generality, assume that $B_{1/n}(q) \sqcap B_{1/m}(p)$ lives on 1. Then $(q_i - 1/n_j, q_i + 1/n_j) \cap (p_k - 1/m_l, p_k + 1/m_l) \neq \emptyset$ for all i, j, k and l such that $a_i \wedge b_j \wedge c_k \wedge d_l > 0$. Thus, for every i, j, k, l and s with $a_i \wedge b_j \wedge c_k \wedge d_l \wedge e_t > 0$ there exists $s_{ijklt} \in \mathbb{N}$ such that

$$\Big(x_t - \frac{1}{s_{ijklt}}, x_t + \frac{1}{s_{ijklt}}\Big) \subseteq \Big(q_i - \frac{1}{n_j}, q_i + \frac{1}{n_j}\Big) \cap \Big(p_k - \frac{1}{m_l}, p_k + \frac{1}{m_l}\Big).$$

Then $B_{1/s}(x) \sqsubseteq B_{1/n}(q) \sqcap B_{1/m}(p)$ where $s = \sum (a_i \wedge b_j \wedge c_k \wedge d_l \wedge e_t) s_{ijklt}$. Since $\sqcup \mathcal{B} = \mathbf{Q}$, it holds that $\mathcal{B} \in \mathcal{S}(\mathcal{S}(\mathbf{Q}))$ is a conditional topological base, due to Lemma 4.1.4. By Lemma 3.4.6, $\mathcal{B}$ is conditionally countable, and thus $\mathbf{Q}$ endowed with $\mathcal{T}^{\mathcal{B}}$ is conditionally separable. To see the conditional Hausdorff property, let $q, p \in \mathbf{Q}$ with $q = \sum a_i q_i$ and $p = \sum b_j p_j$ be such that $q \sqcap p = \mathbf{0}$. Then $p_j \neq q_i$ for all i, j with $a_i \wedge b_j > 0$. Choose $n_{ij} \in \mathbb{N}$ such that $0 < 1/n_{ij} < |p_j - q_i|/2$ for every i, j with $a_i \wedge b_j > 0$. Then $B_{1/n}(q) \sqcap B_{1/n}(p) = \mathbf{0}$ where $n = \sum (a_i \wedge b_j) n_{ij}$. $\square$

Definition 5.2.3. The conditional topology $\mathcal{T}^{\mathcal{B}}$ on $\mathbf{Q}$ is called the *conditional Euclidean topology* of $\mathbf{Q}$. A conditional sequence $(q_n) \subseteq \mathbf{Q}$ is said to be *conditionally Cauchy* if for every $N \in \mathbf{N}$ there exists $n_0 \in \mathbf{N}$ such that $|q_n - q_m| < 1/N$ for all $m, n \geqslant n_0$. Denote by $\mathcal{C}$ the collection of all conditional Cauchy sequences in $\mathbf{Q}$.

Lemma 5.2.4. *Every $(q_n) \in \mathcal{C}$ is conditionally bounded.*

Proof. Let (q_n) be a conditional Cauchy sequence in $\mathbf{Q}$. Then there exists $n_0 \in \mathbf{N}$ such that $|q_n - q_m| < 1$ for all $n, m \geqslant n_0$. By the conditional triangular inequality, it holds

$$|q_m| = |q_{n_0} - q_m + q_{n_0}| \leqslant |q_{n_0} - q_m| + |q_{n_0}| \leqslant 1 + |q_{n_0}|, \quad m \geqslant n_0.$$

Thus, (q_n) is conditionally bounded by $1 + \max\{|q_n| : 1 \leqslant n \leqslant n_0\}$. □

Proposition 5.2.5. *The conditional Cauchy sequences $\mathcal{C}$ are a conditional subset of $\mathbf{Q}^{\mathbf{N}}$ and $(\mathcal{C}, +, \cdot)$ is a conditional subring of $(\mathbf{Q}^{\mathbf{N}}, +, \cdot)$.*

Proof. First, we verify that $\mathcal{C}$ is a conditional subset of $\mathbf{Q}^{\mathbf{N}}$ by showing that it is $\mathcal{A}$-stable. Let (a_i) be a partition of unity, $(q^i) \subseteq \mathcal{C}$, and $N \in \mathbf{N}$ with $N = \sum b_j n_j$. For every i and j with $a_i \wedge b_j > 0$ there exists $m_{ij} \in \mathbf{N}$ such that $|q^i_n - q^i_m| < 1/n_j$ for all $m, n \geqslant m_{ij}$. Then

$$\left|\sum a_i(q^i_n - q^i_m)\right| = \sum a_i \left|q^i_n - q^i_m\right| < \frac{1}{N}, \quad n, m \geqslant m_0,$$

where $m_0 := \sum(a_i \wedge b_j) m_{ij}$.

Secondly, we show that $\mathcal{C}$ is closed under conditional addition and conditional multiplication. Let $(q_n), (p_n) \in \mathcal{C}$ and $N \in \mathbf{N}$. Since every conditional Cauchy sequence is conditionally bounded, due to Lemma 5.2.4, there exists $B \in \mathbf{N}$ such that

$$|q_n| \leqslant B, \quad |p_n| \leqslant B, \qquad n \in \mathbf{N}.$$

Set $M := 2BN \in \mathbf{N}$. Then there exists $n_0 \in \mathbf{N}$ such that

$$|q_n - q_m| < \frac{1}{M}, \quad |p_n - p_m| < \frac{1}{M}, \qquad m, n \geqslant n_0.$$

By the conditional triangular inequality, we obtain the following conditional bounds

$$|q_n + p_n - (q_m + p_n)| \leqslant |q_n - q_m| + |p_n - p_m| < \frac{2}{M} \leqslant \frac{1}{N}, \quad m, n \geqslant n_0,$$

and

$$|q_n p_n - q_m p_m| \leqslant |p_n|\,|q_n - q_m| + |p_n - p_m|\,|q_m| < \frac{2B}{M} = \frac{1}{N}, \quad m, n \geqslant n_0.$$

Hence, $(q_n + p_n)$ and $(q_n p_n)$ belong to $\mathcal{C}$. Thus, $(\mathcal{C}, +, \cdot)$ is a conditional subring of $(\mathbf{Q}^{\mathbf{N}}, +, \cdot)$. □

Let $(q_n) \sim (p_n)$ if $(q_n - p_n) \to 0$ for some pair $(q_n), (p_n) \in \mathcal{C}$. Then $\sim$ is a conditional equivalence relation on $\mathcal{C}$. Indeed, by an argument similar as in the first part of the proof of Proposition 5.2.5, one shows that $\sim$ is a conditional relation. The properties of conditional reflexivity and symmetry follow from the symmetry of the conditional addition, and the conditional transitivity is guaranteed by the conditional triangular inequality. Define on $\mathbf{R} := \mathcal{C}/\sim$ the following operations

$$[(q_n)] + [(p_n)] = [(q_n) + (p_n)], \quad [(q_n)][(p_n)] = [(q_n)(p_n)].$$

Then one can show by a similar argument as in the second part of the proof of Proposition 5.2.5 that $+$ and $\cdot$ are well-defined. The commutativity, associativity and distributivity of $+$ and $\cdot$ and that $[(0)]$ and $[(1)]$ are the neutral elements follow from the respective properties in $(\mathcal{C}, +, \cdot)$. The conditional triangular inequality implies that $[(-q_n)]$ is the inverse of $[(q_n)]$. Thus, $(\mathbf{R}, +, \cdot)$ is a conditional ring. Define on $\mathbf{R}$ the conditional relation $[(p_n)] \leqslant [(q_n)]$ if and only if there exist $a \in \mathcal{A}, N \in a\mathbf{N}$ and $n_0 \in a\mathbf{N}$ such that

$$aq_n - ap_n > \frac{1}{N}, \quad n \geqslant n_0, \quad \text{and} \quad (a^c p_n) \sim (a^c q_n).$$

Lemma 5.2.6. *If $(q_n) \in \mathcal{C}$ has a conditionally convergent subsequence, then (q_n) is conditionally convergent.*

Proof. Let (q_{n_k}) be a conditionally convergent subsequence of (q_n) and denote by $q \in \mathbf{Q}$ its conditional limit. For any $N \in \mathbf{N}$ there exists $n_0 \in \mathbf{N}$ such that $|q_n - q_m| < 1/(2N)$ for all $n, m \geqslant n_0$. Let $m_0 \in \mathbf{N}$ be such that $|q_{n_k} - q| < 1/(2N)$ for all $n_k \geqslant m_0$. Pick some $n_k \geqslant \max\{n_0, m_0\}$. Then it holds

$$|q_n - q| \leqslant |q_n - q_{n_k}| + |q_{n_k} - q| < \frac{1}{N}, \quad n \geqslant n_0.$$

Hence, (q_n) conditionally converges to q. $\square$

Theorem 5.2.7. *The structure $(\mathbf{R}, +, \cdot, \leqslant)$ is a conditionally ordered field. Moreover, $(\mathbf{Q}, +, \cdot, \leqslant)$ is isomorphic to a conditionally ordered subfield of $(\mathbf{R}, +, \cdot, \leqslant)$.*

Proof. (Step 1) We have to verify that $\leqslant$ is independent of the choice of representatives. Let $[(p_n)], [(q_n)] \in \mathbf{R}$ and assume, without loss of generality, that there exist $N, n_0 \in \mathbf{N}$ such that $q_n - p_n > 1/N$ for all $n \geqslant n_0$. Let $(p'_n) \sim (p_n)$ and $(q'_n) \sim (q_n)$. Define

$$H = \{a \in \mathcal{A} : \exists\, M, m \in a\mathbf{N} \text{ such that } aq'_n - ap'_n > \frac{1}{M} \text{ for all } n \geqslant m\}.$$

If $a_* := \vee H = 1$, then there exists a partition of unity (a_i) such that for all a_i there is some $b_i \in H$ and $a_i \leq b_i$, by Assumption **(P)**. By $\mathcal{A}$-stability, there exist $M_0, m_0 \in \mathbf{N}$ such that $q'_n - p'_n > 1/M_0$ for all $n \geq m_0$. Now assume, for the sake of contradiction, that $0 < a_*^c$. For every $N, n \in \mathbf{N}$, let

$$b_{N,n} := \vee\{a \in \mathcal{A}_{a_*^c} : aq'_m - ap'_m \leqslant 1/N \text{ for some } m \geqslant n\}.$$

By assumption, $\wedge b_{N,n} > 0$. Indeed, if $\wedge b_{N,n} = 0$, then $\vee b_{N,n}^c \wedge a_*^c = a_*^c$, by de Morgan's law. In this case there exists a partition $(a_i) \in \mathcal{K}(a_*^c)$ such that $a_i \leq b_{N_i,n_i}^c$ for each i, due to Assumption **(P)**. However, $\mathcal{A}$-stability then implies that $a_*^c q'_m - a_*^c p'_m > 1/N$ for all $m \geq n$, where $N = \sum a_i N_i$ and $n = \sum a_i n_i$, which contradicts the maximality of a_*. Thus, $b_* := \wedge b_{N,n} > 0$. Then there exists a conditional subsequence of $(b_* q'_n - b_* p'_n)$ conditionally converging to 0. By Lemma 5.2.6, $(b_* q'_n - b_* p'_n)$ conditionally converges to 0. By the conditional symmetry of conditional equivalence relations, it holds $(b_* q_n) \sim (b_* p_n)$ which contradicts the assumptions. Hence, $a = 1$.

(Step 2) We show that $\leqslant$ is a conditional total order. Let (a_i) be a partition of unity and $[(q_n^i)], [(p_n^i)] \subseteq \mathbf{R}$ be such that $[(p_n^i)] \leqslant [(q_n^i)]$ for every $i \in I$. For each $i \in I$, there exist $b_i \in \mathcal{A}$, $N_i \in b_i\mathbf{N}$ and $n_i \in b_i\mathbf{N}$ such that

$$b_i q_n^i - b_i p_n^i > 1/N_i, \quad n \geqslant n_i, \quad \text{and} \quad (b_i^c p_n^i) \sim (b_i^c q_n^i).$$

For $a := \vee(a_i \wedge b_i)$, it holds

$$a \sum a_i q_n^i - a \sum a_i p_n^i > \frac{1}{N}, \quad n \geqslant n_0, \quad \text{and} \quad \left(a^c \sum a_i q_n^i\right) \sim \left(a^c \sum a_i p_n^i\right),$$

where $N = \sum (a_i \wedge b_i) N_i$ and $n_0 = \sum (a_i \wedge b_i) n_i$. Hence, $\leqslant$ is a conditional relation. The conditional reflexivity of $\leqslant$ follows from the conditionally reflexivity of $\sim$, and its conditional transitivity follows from the conditional triangular inequality. As for the conditional antisymmetry, let $[(p_n)] \leqslant [(q_n)]$ and $[(q_n)] \leqslant [(p_n)]$. Then there exist $a, b \in \mathcal{A}$, $N, n_0 \in a\mathbf{N}$ and $M, m_0 \in b\mathbf{N}$ such that

$$aq_n - ap_n > \frac{1}{N}, \quad n \geqslant n_0 \quad \text{and} \quad bp_n - bq_n > \frac{1}{M}, \quad m \geqslant m_0.$$

Since $(a^c q_n) \sim (a^c p_n)$ and $(b^c q_n) \sim (b^c p_n)$, it holds that $(a^c \vee b^c q_n) \sim (a^c \vee b^c p_n)$ by the $\mathcal{A}$-stability of $\leqslant$. Hence, $a = b = 0$, and thus $(q_n) \sim (p_n)$. To see that $\leqslant$ is conditionally total, let $[(q_n)], [(p_n)] \in \mathbf{R}$ and define

$$\begin{aligned} a &= \vee\{d \in \mathcal{A} : (dp_n) \sim (dq_n)\}, \\ b &= \vee\{d \in \mathcal{A} : \exists\, N, n_0 \in d\mathbf{N} \text{ such that } dp_n - dq_n > \frac{1}{N} \text{ for all } n \geqslant n_0\}, \\ c &= \vee\{d \in \mathcal{A} : \exists\, N, n_0 \in d\mathbf{N} \text{ such that } dq_n - dp_n > \frac{1}{N} \text{ for all } n \geqslant n_0\}. \end{aligned}$$

Obviously, a, b and c are pairwise disjoint. Assume, for the sake of contradiction, that $d := a \vee b \vee c < 1$. Due to the maximality of b and c and $\mathcal{A}$-stability, for all $N, n_0 \in d^c\mathbf{N}$ there exist $n, m \geqslant n_0$ such that $d^c p_n - d^c q_n \leqslant 1/N$ and $d^c q_n - d^c p_n \leqslant 1/N$. Hence, there exists a conditional subsequence $(d^c q_n - d^c p_n)$ which conditionally converges to 0. By

Lemma 5.2.6, $(d^c q_n) \sim (d^c p_n)$ which, however, contradicts the maximality of a. Thus, $a \vee b \vee c = 1$.

(Step 3) We have already seen that $(\mathbf{R}, +, \cdot)$ is a conditional ring. Let $[(p_n)]$ and $[(q_n)]$ be in $\mathbf{R}$ with $[(q_n)] < [(p_n)]$. Then there exist $N, n_0 \in \mathbf{N}$ such that $p_n - q_n > 1/N$ for all $n \geqslant n_0$. For any $[(r_n)]$ in $\mathbf{R}$, it follows that $(p_n + r_n) - (q_n + r_n) = p_n - q_n > 1/N$ for all $n \geqslant n_0$. Hence, $[(q_n)] + [(r_n)] < [(p_n)] + [(r_n)]$. If $[(q_n)], [(p_n)] > 0$, then there exist $N, M, n_0, m_0 \in \mathbf{N}$ such that $q_n > 1/N$ for all $n \geqslant n_0$ and $p_n > 1/M$ for all $n \geqslant m_0$. It holds $q_n p_n > 1/MN$ for all $n \geqslant \max\{n_0, m_0\}$ since $\mathbf{Q}$ is a conditionally ordered field. Thus, $(\mathbf{R}, +, \cdot, \leqslant)$ is a conditionally ordered ring. To show that $\mathbf{R}$ is a conditional field, let $[(q_n)] \in [(0)]^{\sqsubset}$. We may assume that $[(q_n)] > 0$. Then there exists $N, n_0 \in \mathbf{N}$ such that $q_n \geqslant 1/N$ for all $n \geqslant n_0$. Let $p_n := a1/q_n + a^c 0$ for each $n \in \mathbf{N}$ where $a \in \mathcal{A}$ is such that $an \geqslant an_0$ and $a^c n < a^c n_0$. Since (q_n) is conditionally Cauchy, we find for every $M \in \mathbf{N}$ some $n_1 \geqslant n_0$ such that $|q_n - q_m| < 1/(MN^2)$, for all $n, m \geqslant n_1$. This implies

$$|p_n - p_m| = \left| \frac{q_n - q_m}{q_n q_m} \right| \leqslant N^2 \, |q_n - q_m| < \frac{1}{M}, \quad m, n \geqslant n_1.$$

Thus, $(p_n) \in \mathcal{C}$. Since it holds $[(p_n)][(q_n)] = [(p_n q_n)] = [(1)]$, it follows that $[(p_n)]$ is the conditional inverse of $[(q_n)]$.

(Step 4) We embed $\mathbf{Q}$ in $\mathbf{R}$ by mapping each $q \in \mathbf{Q}$ to $[(q)] \in \mathbf{R}$. Then

$$\begin{aligned} p + q &\longmapsto [(p+q)] = [(p)] + [(q)], \\ pq &\longmapsto [(pq)] = [(p)][(q)], \end{aligned}$$

and if $[(p)] \leqslant [(q)]$, then there exists $a \in \mathcal{A}$ such that $ap < aq$ and $a^c p = a^c q$ which implies $p \leqslant q$. $\square$

Lemma 5.2.8. *Every conditionally increasing (decreasing) sequence $(q_n) \in \mathbf{Q}^{\mathbf{N}}$ which is conditionally bounded from above (from below) is conditionally Cauchy.*

Proof. For the sake of contradiction, suppose there exist $a > 0$ and $N \in a\mathbf{N}$ such that for all $n_0 \in a\mathbf{N}$ there are $n, m \geqslant n_0$ with $|aq_n - aq_m| > 1/N$ for some $n, m \geqslant n_0$. Since (q_n) is conditionally increasing, the conditional sequence (q_n) exceeds all conditional bounds on a. However, this contradicts the assumptions. Hence, (q_n) is a conditional Cauchy sequence. Similarly, one shows that a conditionally decreasing sequence conditionally bounded from below is conditionally Cauchy. $\square$

Lemma 5.2.9. *Every conditionally increasing (decreasing) sequence $(\rho_n) \subseteq \mathbf{R}$ conditionally bounded from above (from below) has a conditional supremum (infimum) in* $\mathbf{R}$.

Proof. Let (ρ_n) be conditionally increasing and bounded from above. If there exists $k \in \mathbf{N}$ such that $\rho_n = \rho_k$ for all $n \geqslant k$, then $\sup\{\rho_n : n \in \mathbf{N}\} = \rho_k$. If this is not the case, let $k_1 := 1$ and $a_1 := 1$ and define

$$a_2 = \vee\{b \in \mathcal{A} : \exists\, k \in b\mathbf{N} \text{ such that } b\rho_{k_1} < b\rho_k\}.$$

By $\mathcal{A}$-stability, a_2 is attained. Denote by $k_2 = k \in \mathbf{N}$ such that $a_2\rho_{k_1} < a_2\rho_k$. By assumption, $a_2 > 0$. Define recursively a sequence $(a_n) \subseteq \mathcal{A}$ such that

$$a_n = \vee\{b \in \mathcal{A}_{a_{n-1}} : \exists\, k \in b\mathbf{N} \text{ such that } b\rho_{k_{n-1}} < b\rho_k\},$$

and denote by $k_n = k \in \mathbf{N}$ such that $a_n\rho_{k_{n-1}} < a_n\rho_k$. Then $\wedge a_n > 0$, since otherwise there exists a partition of unity (b_i) with $b_i \leq a_{n_i}^c$ and such that $\rho_n = \rho_{\sum b_i k_{n_i}}$ for all $n \geqslant \sum b_i k_{n_i}$, due to De Morgan's law and Assumption **(P)**. Assume, without loss of generality, that $\wedge a_n = 1$. Then

$$\rho_{k_1} < \rho_{k_2} < \ldots < \rho_{k_n} < \ldots\ .$$

Every ρ_{k_n} is of the form $[(r_m^{k_n})] \in \mathbf{R}$. For every $n \in \mathbb{N}$, we find $l_n, L_n \in \mathbf{N}$ such that

$$r_m^{k_{n+1}} - r_m^{k_n} > \frac{1}{L_n}, \quad m \geqslant l_n.$$

We can choose the sequence (l_n) such that $l_n \leqslant l_{n+1}$. Since $(r_m^{k_n})$ and $(r_m^{k_{n+1}})$ are conditionally Cauchy, we find $m_n \geqslant l_n$ such that

$$r_m^{k_n} - r_{m_n}^{k_n} < \frac{1}{4L_n}, \quad r_{m_n}^{k_{n+1}} - r_m^{k_{n+1}} < \frac{1}{4L_n}, \quad m \geqslant m_n.$$

Then for $s_n := r_{l_n}^{k_n} + 1/(2L_n)$ it holds

$$r_m^{k_{n+1}} - s_n > \frac{1}{4L_n}, \quad s_n - r_m^{k_n} > \frac{1}{4L_n}, \quad m \geqslant m_n.$$

Hence,

$$\rho_{k_n} = [(r_m^{k_n})] < [(s_n)] = s_n[(1)] < [(r_m^{k_{n+1}})] = \rho_{k_{n+1}}, \quad n \in \mathbb{N}. \tag{5.2.1}$$

For every partition of unity (c_i) and every $(k_{n_i}) \subseteq \mathbb{N}$ define $\rho_{\sum c_i k_{n_i}} = \sum c_i \rho_{k_{n_i}}$. We obtain a conditional subsequence (ρ_{k_n}) of (ρ_n) which by construction is conditionally strictly increasing. Since (ρ_n) is conditionally increasing, the conditional supremum of (ρ_{k_n}) coincides with the one of (ρ_n). For every partition of unity (c_i) and every $(n_i) \subseteq \mathbb{N}$ define $s_{\sum c_i n_i} = \sum c_i s_{n_i}$. We show that $s = (s_n)$ is the conditional supremum of (ρ_{k_n}). By construction, (s_n) is conditionally increasing in $\mathbf{Q}$. Since (ρ_n) is conditionally bounded from above, (s_n) is conditionally bounded from above, due to (5.2.1). By Lemma 5.2.8, it holds $[(s_n)] \in \mathbf{R}$. By (5.2.1), it further holds $\rho_{k_n} \leqslant [(s_n)]$ for all $n \in \mathbf{N}$. If there exists $\rho \in \mathbf{R}$ such that $\rho_{k_n} \leqslant \rho < [(s_n)]$ for all $n \in \mathbf{N}$, then

$$[(s_n)] < \rho_{k_{n+1}} \leqslant \rho < [(s_n)], \quad n \in \mathbf{N},$$

which is absurd. Thus, $[(s_n)] = \sup\{\rho_n : n \in \mathbf{N}\}$.

Similarly, one shows that a conditionally decreasing sequence which is conditionally bounded from below has a conditional infimum. □

Proposition 5.2.10. *Let $Y \in \mathcal{S}(\mathbf{R})$ be conditionally bounded from above (from below), then Y has a conditional supremum (infimum).*

Proof. Let $\rho \in \mathbf{R}$ be a conditional upper bound for Y. We construct a conditionally increasing sequence (α_n) and a conditionally decreasing sequence (β_n) as follows. Choose $\alpha_1 \in Y$ arbitrarily and set $\beta_1 := \rho$ and $\rho_1 := (\alpha_1 + \beta_1)/2$. Let

$$a := \vee\{b \in \mathcal{A} : \exists\, y \in Y \text{ with } by \geqslant \rho_1\}.$$

Set $\alpha_2 := a\rho_1 + a^c\alpha_1$, $\beta_2 := a\beta_1 + a^c\rho_1$ and $\rho_2 := (\alpha_2 + \beta_2)/2$. Repeat the previous step to obtain two sequences (α_n) and (β_n). Let $\alpha_{\sum c_i n_i} := \sum c_i \alpha_{n_i}$ and $\beta_{\sum c_i n_i} := \sum c_i \beta_{n_i}$ in order to obtain two conditional sequences (α_n) and (β_n). By construction, (α_n) is conditionally increasing and (β_n) conditionally decreasing. Moreover, it holds

$$0 \leqslant \beta_n - \alpha_n \leqslant \frac{\beta_1 - \alpha_1}{\sum c_i 2^{n_i}}, \tag{5.2.2}$$

where $n = \sum c_i n_i$. Since (α_n) is conditionally bounded from above by ρ and (β_n) is conditionally bounded from below by α_1, there exist $\alpha := \sup\{\alpha_n : n \in \mathbf{N}\}$ and $\beta := \inf\{\beta_n : n \in \mathbf{N}\}$, due to Lemma 5.2.9. By (5.2.2), it holds

$$0 \leqslant \beta - \alpha \leqslant \inf\left\{\frac{\beta_1 - \alpha_1}{\sum c_i 2^{n_i}} : (c_i) \in \mathcal{K}(1), (n_i) \subseteq \mathbb{N}\right\} = 0.$$

Hence, $\alpha = \beta$. By construction, $y \leqslant \beta_n$ for all $y \in Y$ and $n \in \mathbf{N}$. This implies

$$y \leqslant \inf\{\beta_n : n \in \mathbf{N}\} = \beta = \alpha = \sup\{\alpha_n : n \in \mathbf{N}\} \leqslant \rho, \quad y \in Y.$$

Since the conditional upper bound ρ was chosen arbitrarily, it follows $\alpha = \sup(Y)$.

Similarly, one proves that every $Y \in \mathcal{S}(\mathbf{R})$ which is conditionally bounded from below has a conditional infimum. □

Definition 5.2.11. We call $\mathbf{R} = (\mathbf{R}, +, \cdot, \leqslant)$ the *conditionally totally ordered field of the conditional real numbers.*[1] Henceforth, we denote its elements by x, y, z.

We state the conditional Archimedean principle.

Lemma 5.2.12. *The conditional natural numbers are not conditionally bounded in $\mathbf{R}$, that is for every $x \in \mathbf{R}$ there exists $n \in \mathbf{N}$ such that $n > x$.*

[1] Note that $\mathbf{R}$ is not the conditional set generated by $\mathbb{R}$ unless $\mathcal{A}$ is trivial.

Proof. Fix $x \in \mathbf{R}$. Let $a = \vee\{b \in \mathcal{A} : bx < b1\}$. By $\mathcal{A}$-stability, a is attained and x can be bounded on a by 1. On a^c, let

$$A = \{n \in \mathbf{N} : a^c n \leqslant a^c x\} \sqsubseteq \mathbf{R}.$$

Since A is conditionally bounded, it has a conditional supremum $a^c s$ in $a^c\mathbf{R}$, due to Proposition 5.2.10. Hence, there exists $n \in A$ such that $a^c s - a^c 1/2 < a^c n$ which implies $a^c(n+1) > a^c s$. Since $a^c x - a^c s \leqslant 1$, there exists a conditional natural number n for which $a^c n \leqslant a^c x$ is false. Define

$$c = \vee\{b \leq a^c : bn > bx \text{ for some } n \in \mathbf{N}\}.$$

By $\mathcal{A}$-stability, c is attained by some $n_x \in \mathbf{N}$. If we suppose that $c < a^c$, the previous argument on c^c yields a contradiction. Hence, $c = a^c$ and we can conditionally bound x by $a1 + a^c n_x$. $\square$

Let $\mathbf{R}_{++} := \{x \in \mathbf{R} : 0 < x\}$. By the conditional Archimedean Principle, for every $\varepsilon \in \mathbf{R}_{++}$ there exists $n \in \mathbf{N}$ such that $0 < 1/n < \varepsilon$.

Proposition 5.2.13. *For each $\varepsilon \in \mathbf{R}_{++}$ and $x \in \mathbf{R}$, let*

$$B_\varepsilon(x) := \{y \in \mathbf{R} : |x - y| < \varepsilon\}.$$

Then $\mathcal{B} = \{B_\varepsilon(x) : x \in \mathbf{R}, \varepsilon \in \mathbf{R}_{++}\}$ is a conditional topological base of a conditionally Hausdorff topology, and $\mathbf{R}$ endowed with $\mathcal{T}^{\mathcal{B}}$ is conditionally separable.

Proof. By the $\mathcal{A}$-stability of $\leqslant$, $\mathbf{R}$ and $\mathbf{R}_{++}$, it holds $\mathcal{B} \in \mathcal{S}(\mathcal{S}(\mathbf{R}))$. Let $B_{\varepsilon_1}(x_1), B_\varepsilon(x_2) \in \mathcal{B}$ and

$$x \in B_{\varepsilon_1}(x_1) \sqcap B_\varepsilon(x_2) = a_* B_{\varepsilon_1}(x_1) \cap a_* B_\varepsilon(x_2) := A.$$

Without loss of generality, assume $a_* = 1$. Note that A is conditionally bounded. By Proposition 5.2.10, A has a conditional infimum $\alpha := \inf(A)$ and a conditional supremum $\beta := \sup(A)$. Set $\delta := (\min\{x - \inf(A), \sup(A) - x\})/2$. Then $\delta \in \mathbf{R}_{++}$ and $B_\delta(x) \sqsubseteq B_{\varepsilon_1}(x_1) \sqcap B_\varepsilon(x_2)$. Since $\sqcup\mathcal{B} = \mathbf{R}$, it holds that $\mathcal{B}$ is a conditional topological base, due to Lemma 4.1.4. To see that $\mathcal{T}^{\mathcal{B}}$ is conditionally Hausdorff, let $x, y \in \mathbf{R}$ such that $x \sqcap y = \mathbf{0}$. Since $\mathbf{R}$ is conditionally totally ordered, there exists $a \in \mathcal{A}$ such that $ax > ay$ and $a^c y > a^c x$. By the conditional Archimedean Principle there exist $N, M \in \mathbf{N}$ such that $ax - ay > a\frac{1}{N}$ and $a^c y - a^c x > a^c \frac{1}{M}$. Setting $\delta := a1/(2N) + a^c 1/(2M)$, yields $B_\delta(x) \sqcap B_\delta(y) = \mathbf{0}$. Let $x \in \mathbf{R}$ and $n \in \mathbf{N}$. It holds $x + 1/(2n) \in B_{1/n}(x)$ and $x < x + 1/(2n)$. By the same argument as in the proof of Lemma 5.2.9, one constructs some $s \in \mathbf{Q}$ such that $x < s < x + 1/(2n)$. By the conditional Archimedean principle, for every $\varepsilon \in \mathbf{R}_{++}$, we find some $s \in \mathbf{Q}$ such that $s \in B_\varepsilon(x)$. By the characterization of the conditional closure in (4.1.3), $cl(\mathbf{Q}) = \mathbf{R}$, since x and ε were chosen arbitrarily. Thus, $\mathbf{R}$ is conditionally separable. $\square$

Definition 5.2.14. The conditional topology generated by $\mathcal{B}$ on $\mathbf{R}$ is called the *conditional Euclidean topology* of $\mathbf{R}$.

We prove the conditional version of the Peak Point Lemma.

Lemma 5.2.15. *Let (x_n) be a conditional sequence of conditional real numbers. Then (x_n) has a conditionally monotone subsequence.*

Proof. Let

$$a := \vee\{b \in \mathcal{A} : \{bx_n : n > m\} \text{ does not have a conditional max for some } m \in b\mathbf{N}\}.$$

We notice that a is attained by some $m_0 \in a\mathbf{N}$. Furthermore, for each ax_m with $m > m_0$ there exists $n > m$ such that $ax_n > ax_m$. By construction, $\{a^c x_n : n > m\}$ has a conditional maximum for all $m \in a^c\mathbf{N}$. Let

$$x_{n_1} := a^c \max_{n>1}(a^c x_n) + ax_{m_1}, \quad x_{n_2} := a^c \max_{n>n_1}(a^c x_n) + ax_{m_2}, \ \ldots,$$

where $ax_{m_1} := ax_{m_0+1}$ and ax_{m_i} is chosen such that $ax_{m_i} < ax_{m_{i+1}}$ for $i \geq 2$. Set $x_{\sum a_i n_{r_i}} := \sum a_i x_{n_{r_i}}$ for every $(x_{n_{r_i}})$ and every partition of unity (a_i). By construction, (x_{n_r}) is a conditionally monotone subsequence of (x_n). □

Lemma 5.2.16. *Every conditionally monotone and bounded sequence (x_n) is conditionally convergent.*

Proof. Let $a \in \mathcal{A}$ be such that (ax_n) is conditionally increasing and bounded from above and $(a^c x_n)$ is conditionally decreasing and bounded from below. Let x^s be the conditional supremum of $\{ax_n : n \in \mathbf{N}\}$ and x^i be the conditional infimum of $\{a^c x_n : n \in \mathbf{N}\}$, due to Lemma 5.2.9. We show that $x := ax^s + a^c x^i$ is the conditional limit of (x_n). For every $N \in \mathbf{N}$, we can find x_{n_s} and x_{n_i} such that

$$x^s - a\frac{1}{N} \leqslant ax_{n_s}, \qquad a^c x_{n_i} \leqslant x^i + a^c\frac{1}{N}.$$

Hence,

$$|x - x_n| \leqslant \frac{1}{N}, \quad n \geqslant an_s + a^c n_i.$$

□

The conditional version of the Bolzano-Weierstraß Theorem follows now readily from Lemma 5.2.15 and 5.2.16:

Theorem 5.2.17. *Every conditionally bounded sequence in $\mathbf{R}$ has a conditionally convergent subsequence.*

Finally, we show that $\mathbf{R}$ is conditionally complete.

Proposition 5.2.18. *Every conditional Cauchy sequence in $\mathbf{R}$ is conditionally convergent.*

Proof. Let (x_n) be a conditional Cauchy sequence. Then there exists $n_0 \in \mathbf{N}$ such that $|x_n - x_m| < 1$ for all $n, m \geqslant n_0$. By the conditional triangular inequality, it holds

$$|x_m| = |x_{n_0} - x_m + x_{n_0}| \leqslant |x_{n_0} - x_m| + |x_{n_0}| \leqslant 1 + |x_{n_0}|, \quad n \geqslant n_0.$$

Hence, (x_n) is conditionally bounded by $1 + \max\{|x_n| : n \leqslant n_0\}$. By the conditional Bolzano-Weierstraß Theorem, there exists a conditionally converging subsequence (x_{n_r}). Denote its conditional limit by x and let $\varepsilon > 0$. Choose $n_0, m_0 \in \mathbf{N}$ such that

$$|x - x_{n_r}| < \frac{\varepsilon}{2}, \quad n_r > n_0, \quad |x_n - x_m| < \frac{\varepsilon}{2}, \quad n, m \geqslant m_0.$$

By the conditional triangular inequality, it holds

$$|x_n - x| \leqslant |x_n - x_{n_r}| + |x_{n_r} - x| < \varepsilon, \quad n \geqslant m_0,$$

for some $n_r > n_0$. □

5.3 Connection to L^0-theory

An important example of a conditional set is the one associated to the set of real-valued measurable functions L^0 on a σ-finite measure space $(\Omega, \mathcal{F}, \mu)$. In this section we prove that L^0 is equivalent to the conditional real numbers with respect to the measure algebra associated to $(\Omega, \mathcal{F}, \mu)$. This result establishes the link to L^0-theory, previously introduced and studied in [FKV09, KV08, CKV12, DKKS13, Gou10, GS11].

We adopt the notations in [CKV12]. Let $(\Omega, \mathcal{F}, \mu)$ be a σ-finite measure space and assume $\mu(\Omega) > 0$. We denote by L^0 the set of all measurable functions $f : \Omega \to \mathbb{R}$, where two of them are identified if they agree a.e. (almost everywhere). In particular, for $f, g \in L^0$, $f = g$, $f < g$ and $f \leq g$ will be understood in the a.e. sense. The set L^0 with the a.e. order is a lattice ordered $\mathbb{R}$-algebra. In particular, L^0 is a ring. We use the notations $\mathcal{F}_+ := \{A \in \mathcal{F} : \mu(A) > 0\}$, $L^0_+ := \{f \in L^0 : f \geq 0\}$, $L^0_{++} := \{f \in L^0 : f > 0\}$, and $\mathcal{F}(\mathbb{Q})$ and $\mathcal{F}(\mathbb{N})$ to denote the set of all $f \in L^0$ with values in $\mathbb{Q}$ and $\mathbb{N}$, respectively. A partition of Ω in $\mathcal{F}$ is a family $(A_n) \subseteq \mathcal{F}_+$ such that $\mu(A_n \cap A_m) = 0$ for $n \neq m$ and $\mu(\cup A_n \,\Delta\, \Omega) = 0$. Since $(\Omega, \mathcal{F}, \mu)$ is σ-finite, every partition of Ω is at most countable. For every partition (A_n) and every family $(f_n) \subseteq L^0$, it holds $\sum 1_{A_n} f_n := \lim_{n\to\infty}(1_{A_1} f_1 + \ldots + 1_{A_n} f_n) \in L^0$. The associated measure algebra $\mathcal{A}$ of $(\Omega, \mathcal{F}, \mu)$ is the Boolean algebra emerging from $\mathcal{F}$ by identifying sets $A, B \in \mathcal{F}$ with $\mu(A \,\Delta\, B) = 0$. The Boolean algebra $\mathcal{A}$ is complete and satisfies the countable chain condition. For the precise definition of $\mathcal{A}$ we refer to Appendix 1.

For every $a \in \mathcal{A}$, define $L^0_a := \{f 1_A : A \in a, f \in L^0\}$ and $\gamma_a : L^0_1 \to L^0_a$ by $f \mapsto f 1_A$. Then $\mathbf{L^0} := (L^0_a, \gamma_a)_{a \in \mathcal{A}}$ is a conditional set on $\mathcal{A}$. Indeed, $L^0_0 = \{0\}$ and for every $[a_i, f_i] \subseteq \mathcal{A} \times \mathbf{L^0}$, it holds

$$\sum a_i f_i := \sum_{i \geq 1} f_i 1_{A_i} \in L^0_1.$$

The collections $Q^0_1 := \{f \in L^0_1 : f \in \mathcal{F}(\mathbb{Q})\}$ and $N^0_1 := \{f \in L^0_1 : f \in \mathcal{F}(\mathbb{N})\}$ are $\mathcal{A}$-stable subsets of L^0_1, and the conditional subsets of $\mathbf{L^0}$ associated to them are denoted by $\mathbf{Q^0}$ and $\mathbf{N^0}$, respectively.

Lemma 5.3.1. *The structure $(\mathbf{L^0}, +, \cdot, \leqslant)$ is a conditionally ordered field.*

Proof. Since $(L^0, +, \cdot, \leq)$ is a ordered ring, $(\mathbf{L^0}, +, \cdot, \leqslant)$ is a conditionally ordered ring. For $f \in 0^\sqsubset$, it holds $f(\omega) \neq 0$ a.e. Then $f^{-1}(\omega) := 1/f(\omega)$ a.e. is the conditional inverse of f. To see totality, let $f, g \in L^0_1$, and observe that $(\{f < g\}, \{g > f\}, \{f = g\})$ is a partition of unity. □

For every $\varepsilon \in L^0_{++}$ and $f \in L^0$, define $B_\varepsilon(f) := \{g \in L^0 : |f - g| < \varepsilon\}$. Then the collection $\mathcal{B} := \{B_\varepsilon(f) : \varepsilon \in L^0_{++}, f \in L^0\}$ is a base of a topology on L^0. This topology was introduced in [FKV09] and is called the *L^0-topology.* For every partition (A_n) of Ω and $(f_n) \subseteq B_\varepsilon(f)$, it holds $\sum 1_{A_n} f_n \in B_\varepsilon(f)$. Moreover, for every $(B_{\varepsilon_n}(f_n)) \subseteq \mathcal{B}$, it holds $\sum 1_{A_n} B_{\varepsilon_n}(f_n) = B_{\sum 1_{A_n}\varepsilon_n}(\sum 1_{A_n} f_n) \in \mathcal{B}$. By Proposition 4.1.5, the L^0-topology can be lifted to a conditional topology on the conditional set $\mathbf{L^0}$, called the *conditional $\mathbf{L^0}$-topology.*

Lemma 5.3.2. *The conditional $\mathbf{L^0}$-topology is conditionally Hausdorff and the conditional topological space $\mathbf{L^0}$ is conditionally separable.*

Proof. We show that
$$\left\{B_{\frac{1}{N}}(Q) : N \in \mathbf{N^0}, Q \in \mathbf{Q^0}\right\}$$
is a conditional base. Let $O \subseteq \mathbf{L^0}$ be conditionally open. For every $x \in O$, there exists $\varepsilon \in \mathbf{L^0_{++}}$ such that $B_\varepsilon(x) \subseteq O$. Let $a_1 := \{\varepsilon > 1\}$ and $a_n := \{\frac{1}{n-1} \leq \varepsilon < \frac{1}{n}\}$, for $n \geq 2$. Then (a_n) is a partition of unity and $0 < \frac{1}{N} < \varepsilon$ where $N = \sum n a_n$. It holds $B_{\frac{1}{N}}(x) \subseteq O$. Now let $\{q_1, q_2, \ldots\}$ be an enumeration of $\mathbb{Q}$ and define $a_{q_n} := \{|x - q_n| < 1/(2N)\}$. Define $b_1 := a_{q_1}$ and $b_n := a_{q_n} \wedge (a_{q_{n-1}})^c$, for $n \geq 2$. Then (b_n) is a partition of unity and $x \in B_{1/2N}(Q) \subseteq B_\varepsilon(x) \subseteq O$ where $Q = \sum b_n q_n$. By a similar argument, one proves the conditional Hausdorff property. □

Let $\mathbf{Q}$ be the conditional rational numbers with respect to $\mathcal{A}$.

Theorem 5.3.3. *The conditionally totally ordered field of conditional real numbers $\mathbf{R}$ with respect to $\mathcal{A}$ is conditionally isomorphic to $\mathbf{L^0}$ and the conditional real line is conditionally homeomorphic to $\mathbf{L^0}$ equipped with the conditional $\mathbf{L^0}$-topology.*

Proof. Clearly, $(\mathbf{Q}, +, \cdot, \leqslant)$ and $(\mathbf{Q^0}, +, \cdot, \leqslant)$ are conditionally isomorphic structures and $\mathbf{Q^0}$ endowed with the conditional relative $\mathbf{L^0}$-topology is conditionally homeomorphic to $\mathbf{Q}$. By Lemma 5.3.2, we know that $\mathbf{Q^0}$ is conditionally dense in $\mathbf{L^0}$. Hence, for every $f \in \mathbf{L^0}$, there exists a conditional Cauchy sequence $(Q_n) \subseteq \mathbf{Q^0}$ conditionally converging to f. By the conditional triangular inequality, for every conditional Cauchy sequence $(P_n) \subseteq \mathbf{Q^0}$ which conditionally converges to f, it holds that $(P_n - Q_n)$ conditionally converges to 0. Conversely, every conditional Cauchy sequence $(P_n) \subseteq \mathbf{Q^0}$ with $(P_n - Q_n) \to 0$ conditionally converges to f. It remains to show that every equivalence class of Cauchy sequences in $\mathbf{Q^0}$ corresponds to some $f \in \mathbf{L^0}$. By [CKV12, Theorem 3.12], every conditional Cauchy sequence in L^0 converges. By Proposition 4.4.4, every conditional Cauchy sequence $(Q_n) \subseteq \mathbf{Q^0}$ conditionally converges to some $f \in \mathbf{L^0}$. Since the conditional $\mathbf{L^0}$-topology is conditionally Hausdorff, f is unique. By the conditional triangular inequality, every conditional Cauchy sequence $(P_n) \subseteq \mathbf{Q^0}$ with $(Q_n - P_n) \to 0$ conditionally converges to f. Since the passages from conditional equivalence classes of conditional Cauchy sequences in $\mathbf{Q^0}$ to elements of $\mathbf{L^0}$ and from elements of $\mathbf{L^0}$ to conditional equivalence classes of conditional Cauchy sequences in $\mathbf{Q^0}$ is reciprocal, it follows that $\mathbf{R}$ is conditionally isomorphic to $\mathbf{L^0}$. By the conditional triangular inequality, if $[(Q_n)]$ and $[(P_n)]$ conditionally converge, then $[(Q_n)] + [(P_n)]$ and $[(Q_n)][(P_n)]$ conditionally converge. Moreover, if $[(Q_n)] \leqslant [(P_n)]$, then there exists $n_0 \in \mathbf{N}$ such that $Q_n \leqslant P_n$ for all $n \geqslant n_0$. Hence, if f and g are the conditional limits of (Q_n) and (P_n), then $f \leqslant g$. □

5.4 Debreu's Gap Lemma

Debreu's Gap Lemma is pivotal to prove existence of continuous order-preserving functions on ordered topological spaces. There exist many proofs of it, for example [Deb54, Deb64], [Bea92], or [BM02]. The following proof of the conditional version of Debreu's Gap Lemma is an adaptation of [Ouw10].

Definition 5.4.1. Let $(\mathbf{R}, \leqslant)$ be the totally ordered field of conditional real numbers. A *conditional interval* in $\mathbf{R}$ is a conditional set $G \in \mathcal{S}(\mathbf{R})$ which is of one of the following forms

$$G = \{x \leqslant z \leqslant y\}, \quad G = \{x < z < y\}, \quad G = \{x < z \leqslant y\}, \quad G = \{x \leqslant z < y\},$$

for some $x, y \in \mathbf{R}$ with $x \leqslant y$. Let $S \in \mathcal{S}(\mathbf{R})$ be such that $S^{\sqsubset}$ lives on 1. The conditional set S is said to be *conditionally degenerate* if $S = \{*\}$. A *conditional lacuna* of S is a conditionally non-degenerate conditional interval G ($aG = a\{*\}$ if and only if $a = 0$) satisfying

(a) $G \sqsubseteq S^{\sqsubset}$;

(b) there exist $l, u \in S$ with $l \leqslant z \leqslant u$ for all $z \in G$.

A *conditional gap* G of S is a conditionally maximal lacuna with respect to the conditional inclusion $\sqsubseteq$ on $\mathcal{S}(\mathbf{R})$.

Note that the conditional set $\{x < z < y\}$ is conditionally open in the conditional Euclidean topology on $\mathbf{R}$.

Theorem 5.4.2. *For every $S \in \mathcal{S}(\mathbf{R})$ such that $S^{\sqsubset}$ lives on 1, there exists a conditionally strictly increasing function $g : S \to \mathbf{R}$ such that all the conditional gaps of $g(S)$ are conditionally open.*

Proof. (Step 1) By Proposition 5.2.13, one finds for every $n \in \mathbf{N}$ and $q \in \mathbf{Q}$ such that $B_{\frac{1}{n}}(q) \sqcap S$ lives on 1 some $z_{n,q} \in B_{\frac{1}{n}}(q) \sqcap S$. By Lemma 3.4.6,

$$Z := cond(\{z_{n,q} : n \in \mathbf{N}, q \in \mathbf{Q}\} \cup \{x, y \in S : \{z \in S : x < z < y\} = \emptyset\})$$

is conditionally countable since for every $x, y \in S$ with $x < y$ there exists $q \in \mathbf{Q}$ such that $x \leqslant q \leqslant y$. Moreover, for all $x, y \in S$ with $x < y$ there exists $z \in Z$ such that $x \leqslant z \leqslant y$. In the next step we construct a conditionally strictly increasing function $f : Z \to \mathbf{R}$.

(Step 2) First, assume that Z is conditionally finite. Let $n \in \mathbf{N}$ with $n = \sum a_i n_i$ be such that $Z = \{z_k : 1 \leqslant k \leqslant n\}$. Then

$$Z = cond\Big(\Big\{ z_{\sum a_i l_i} : 1 \leq l_i \leq n_i, i \in I \Big\} \Big).$$

For each $k = \sum a_i l_i$, it holds $z_{\sum a_i l_i} = \sum a_i z_{l_i}$ where $z_{l_i} = a_i z_k + a_i^c z_{n_0} = z_{a_i l_i + a_i^c n_0}$ for some $z_{n_0} \in Z$ for each i. On each a_i, define on $\{z_1^i, \ldots, z_{n_i}^i\}$ a conditionally strictly increasing function f_i by the following induction. Let $f_i(z_1^i) = 1/2$ and assume that $\{f_i(z_1^i), f_i(z_2^i), \ldots, f_i(z_{l_i}^i)\}$ have been defined such that f_i is conditionally strictly increasing on $\{z_1^i, z_2^i, \ldots, z_{l_i}^i\}$. To define $f_i(z_{l_i+1}^i)$, we distinguish the following cases:

(i) There exists $k_0, k_1 \in \{1, \ldots, l_i\}$ such that $z_{k_0}^i < z_{l_i+1}^i < z_{k_1}^i$:

$$\begin{aligned} f(z_{l_i+1}^i) :=& \Big(\min\{f_i(z_k^i) : z_k^i \geqslant z_{l_i+1}^i, k = 1, \ldots, l_i\} \\ &- \max\{f_i(z_k^i) : z_k^i \leqslant z_{l_i+1}^i, k = 1, \ldots, l_i\} \Big)/2. \end{aligned}$$

(ii) For all $k \in \{1, \ldots, l_i\}$, it holds $z_{l_i+1}^i \leqslant z_k^i$:

$$f_i(z_{l_i+1}^i) := \frac{\min\{f_i(z_k^i) : k = 1, \ldots, l_i\}}{2}.$$

(iii) For all $k \in \{1, \ldots, l_i\}$, it holds $z_{l_i+1}^i \geqslant z_k^i$:

$$f_i(z_{l_i+1}^i) := \frac{\max\{f_i(z_k^i) : k = 1, \ldots, l_i\} + 1}{2}.$$

Next, define $f(z_k) = \sum a_i f_i(z_{l_i}^i)$ for $z_k = \sum a_i z_{l_i}$. For every partition of unity (b_j) and $(z_{k_j}) \subseteq Z$ with $z_{k_j} = \sum a_i z_{l_{ij}}$, define $f(\sum b_j z_{k_j}) = \sum b_j f(z_{k_j})$. By construction, $f : Z \to \mathbf{R}$ is conditionally strictly increasing.

Secondly, let

$$a := \vee\{b \in \mathcal{A} : b'Z \text{ is not conditionally finite for all } b' \leq b\}.$$

We can assume, without loss of generality, that $a = 1$, since otherwise $a^c Z$ is conditionally finite and we proceed on a^c as in the first part. In this case, there exists a conditional bijection $Z \simeq \mathbf{N}^2$. In particular, if $n \sqcap m = \mathbf{0}$, then $z_n \sqcap z_m = \mathbf{0}$. By the previous induction, we can define a conditionally strictly increasing function f on $\{z_1, z_2, \ldots\} \subseteq Z$. By setting $f(\sum a_i z_{n_i}) := \sum a_i f(z_{n_i})$ for every partition of unity (a_i) and every family $z_{n_i} \subseteq \{z_1, z_2, \ldots\}$, we obtain a conditionally strictly increasing function $f : Z \to \mathbf{R}$.

(Step 3) If there exist $X, Y \in \mathcal{S}(Z)$ such that

(i) $X \sqcup Y = Z$;

(ii) for all $x \in X$ and for all $y \in Y$ it holds $x \leqslant y$;

(iii) either aX has not a conditional maximum for all $a > 0$ or aY has not a conditional minimum for all $a > 0$ (or both);

then $\sup_{x\in X} f(x) = \inf_{y \in Y} f(y)$.

Clearly, it holds $\sup_{x\in X} f(x) \leqslant \inf_{y\in Y} f(y)$. Suppose, for the sake of contradiction, that there exists $a > 0$ such that $a \sup_{x\in X} f(x) <_a a \inf_{y\in Y} f(y)$. Assume, without loss of generality, that $a = 1$. Choose $\varepsilon \in \mathbf{R}_{++}$ such that $0 < \varepsilon < \inf_{y\in Y} f(y) - \sup_{x\in X} f(x)$. Choose $x_0 \in X$ and $y_0 \in Y$ such that

$$\sup_X f(x) - \varepsilon \leqslant f(x_0) \leqslant \sup_X f(x) \quad \text{and} \quad \inf_Y f(y) \leqslant f(y_0) \leqslant \inf_Y f(y_0) + \varepsilon.$$

Since $x_0, y_0 \in Z$, there are $n, m \in \mathbf{N}$ such that $x_0 = z_n$ and $y_0 = z_m$. If aX has no conditional maximum for all $a > 0$, there exists $z_k \in X$ such that $z_k > a_0 = z_n$ and $k > n, m$. Similarly, if aY has no conditional minimum for all $a > 0$, there is $z_k \in Y$ such that $z_k < y_0 = z_m$ and $k > n, m$. In either case, we have found $k > n, m$ such that $z_n < z_k < z_m$. Let $k' := \min\{k > n, m : z_n < z_k < z_m\}$. By construction of f, it holds

$$f(z_{k'}) = \frac{f(x_0) + f(y_0)}{2}.$$

Since $\sup_X f(x) - \varepsilon + \inf_Y f(y) \leqslant f(x_0) + f(y_0) \leqslant \sup_X f(x) + \inf_Y f(y) + \varepsilon$, it follows that

$$\frac{\sup_X f(x) + \inf_Y f(y)}{2} - \frac{\varepsilon}{2} \leqslant f(z_{k'}) \leqslant \frac{\sup_X f(x) + \inf_Y f(y)}{2} + \frac{\varepsilon}{2}.$$

Since $\varepsilon > 0$, it follows that $\sup_X f(x) < f(z_{k'}) < \inf_Y f(y)$ which is a contradiction because of $z_{k'} = ax + a^c y$ for some $x \in X$ and $y \in Y$, by Assumption (i).

(Step 4) Define $g : S \to \mathbf{R}$ by $g(x) := \sup_{z\in Z, z\leqslant x} f(z)$. By construction, g is a conditionally strictly increasing extension of f since Z contains all the pairs $x, y \in S$ such that $\{z \in S : x < z < y\} = \emptyset$. Let G be a conditional gap of $g(S)$, $l := \inf G$ and $u := \sup G$, by Proposition 5.2.10. If we show that $l, u \in G^{\sqsubset}$, then G is a conditionally

[2] Assume $\{z_1, z_2, \ldots, z_n\} \subseteq Z$ have been chosen, in a way that if $m, l \in \{1, \ldots, n\}$ with $m \neq l$, then $z_m \sqcap z_l = \mathbf{0}$. Then there exists $z_{n+1} \in Z \setminus \{z_1, z_2, \ldots, z_n\}$ such that $z_{n+1} \sqcap z_m = \mathbf{0}$ for all $m \in \{1, \ldots, n\}$, since otherwise there exists $a > 0$ such that aZ is conditionally finite. Thus, we obtain a sequence (z_n) of pairwise conditionally disjoint elements of Z. Then $Z \simeq cond(\{z_n : n \in N\})$.

open interval. Define $X = \{x \in Z : x \leqslant l\}$ and $Y = \{y \in Z : y \geqslant u\}$. Since $l < u$ (G is conditionally non-degenerate) and X and Y satisfy the Assumptions (i) and (ii) of the previous step, Assumption (iii) is violated. Hence, there exist $0 < a, b \leq 1$ such that aX has a conditional maximum and bY has a conditional minimum. Let a' be the greatest condition $a \in \mathcal{A}$ for which aX has a conditional maximum and b' be the greatest condition $b \in \mathcal{A}$ for which bY has a conditional minimum. By assumption, it holds $a' \wedge b' = 1$, and there exist a conditional maximum x_0 of X and a conditional minimum y_0 of Y. Moreover, $f(x_0) = l$ and $f(y_0) = u$ since G is a conditionally maximal lacuna of S. Thus, $l, u \in G^{\sqsubset}$ which finishes the proof. □

5.5 Compactness in metric spaces

Definition 5.5.1. Let X be a conditional set. A *conditional metric* is a conditional function $d : X \times X \to \mathbf{R}_+$ such that

(i) $d(x, y) = 0$ if and only if $x = y$;

(ii) $d(x, y) = d(y, x)$ for every $x, y \in X$;

(iii) $d(x, z) \leqslant d(x, y) + d(y, z)$ for every $x, y, z \in X$.

The pair (X, d) consisting of a conditional set X and a conditional metric d is called a *conditional metric space.*

Example 5.5.2. For every $n = \sum a_i n_i \in \mathbf{N}$,

$$d(x, y) := \sum a_i \left(\sum_{1 \leq l_i \leq n_i} |x_{l_i} - y_{l_i}|^2 \right)^{1/2}$$

defines a conditional metric on $\mathbf{R}^n := \sum a_i \prod_{k=1}^{n_i} \mathbf{R}$. ◊

Definition 5.5.3. Let (X, d) be a conditional metric space, $x \in X$ and $\varepsilon \in \mathbf{R}_{++}$. We call $B_\varepsilon(x) := \{y \in X : d(x, y) < \varepsilon\}$ the *conditional open ball* with *conditional radius* ε and *conditional center* x.

Similarly to Proposition 5.2.13, one shows that

$$\mathcal{B} = \{B_\varepsilon(x) : x \in X, \varepsilon \in \mathbf{R}_{++}\}$$

is a conditional topological base on (X, d). The conditional topology $\mathcal{T}^{\mathcal{B}}$ is denoted by $\mathcal{T}^d$. Note that $\mathcal{T}^d$ is conditionally Hausdorff by Property *(iii)* of a conditional metric. Moreover, $\mathcal{T}^d$ is conditionally first countable since the collection of conditional balls with conditional radii $1/n$, where $n \in \mathbf{N}$, forms a conditional neighborhood base, due to the conditional Archimedean principle. A conditional topological space $(X, \mathcal{T}^d)$ is conditionally separable if and only if it is conditionally second countable. By choosing an element from each member of the conditional base, one shows that conditional second countability implies conditional separability. Conversely, the collection of conditional balls with radii $1/n$ and around conditional centers belonging to the conditionally countable dense subset is a conditionally countable base of $\mathcal{T}^d$, due to Lemma 3.4.6.

Definition 5.5.4. Let (X,d) be a conditional metric space. A conditional sequence $(x_n) \subseteq X$ is *conditionally Cauchy* if for every $\varepsilon \in \mathbf{R}_{++}$ there exists $n_0 \in \mathbf{N}$ such that $d(x_n, x_m) \leqslant \varepsilon$ for every $n, m \geqslant n_0$. The conditional metric space (X,d) is called *conditionally complete* if every conditional Cauchy sequence conditionally converges and *conditionally sequentially compact* if every conditional sequence $(x_n) \subseteq X$ has a conditional cluster point in X. A conditional set $Y \in \mathcal{S}(X)$ is *conditionally bounded* if there exists $M \in \mathbf{N}$ such that $d(x,y) \leqslant M$ for all $x, y \in Y$ and *conditionally totally bounded* if for every $\varepsilon \in \mathbf{R}_{++}$ there exists a conditionally finite family $(x_k) \subseteq Y$ such that $(B_\varepsilon(x_k))$ is a conditional open covering of Y.

A conditionally totally bounded metric space is conditionally separable. Indeed, for every $N \in \mathbf{N}$ let $(x_k^N)_{1 \leqslant k \leqslant m_N}$ be a conditionally finite family such that $\sqcup B_{1/N}(x_k^N) = X$. Then $\{x_k^N : 1 \leqslant k \leqslant m_N, N \in \mathbf{N}\}$ is a conditionally countable dense subset of X.

Lemma 5.5.5. *Let (X,d) be a conditional sequentially compact metric space. Then for every conditional open covering $(O^i)_{i \in I}$ there exists $\varepsilon \in \mathbf{R}_{++}$ such that for all $x \in X$ there exists $i \in I$ such that $B_\varepsilon(x) \sqsubseteq O^i$.*

This ε is called a *conditional Lebesgue number* of the conditional open covering (O^i) if it exists.

Proof. We prove the lemma by contradiction. Suppose there exists a conditional open covering $(O^i)_{i \in I}$ such that for all $N \in \mathbf{N}$ there exists $x_N \in X$ such that $B_{1/N}(x_N) \sqsubseteq O^i$ is false for all $i \in I$. Then (x_N) is a conditional sequence and since (X,d) is sequentially compact, there exists a conditionally converging subsequence (x_{N_r}). Let us denote its conditional limit by x. Since $(O^i)_{i \in I}$ is a conditional open covering, there exists a partition of unity $(a_j)_{j \in J}$ and a family $(O^{i_j})_{j \in J}$ such that $x = \sum a_j x_j$ where $x_j \in O^{i_j}$ for each $j \in J$. Since (O^i) is a conditional family of conditionally open sets, there exists $N_0 \in \mathbf{N}$ such that $B_{1/N}(x) \sqsubseteq \sum a_j O^{i_j} = O^{\sum a_j i_j}$. Hence, there exists $n_0 \in \mathbf{N}$ such that $x_{N_r} \in B_{1/N}(x)$ for all $N_r \geqslant n_0$. However, this contradicts the assumptions. $\square$

We prove the conditional version of the Borel-Lebesgue Theorem.

Theorem 5.5.6. *Let (X,d) be a conditional metric space. The following assertions are equivalent:*

(i) (X,d) *is conditionally compact;*

(ii) (X,d) *is conditionally complete and totally bounded;*

(iii) (X,d) *is conditionally sequentially compact.*

Proof. (Step 1) To show that *(i)* implies *(ii)*, let $\varepsilon \in \mathbf{R}_{++}$. Since $(B_\varepsilon(x))_{x\in X}$ is a conditional open covering of the conditionally compact set X, there exists a conditionally finite family (x_k) such that $(B_\varepsilon(x_k))$ conditionally covers X. Thus, X is conditionally totally bounded. Let now (x_n) be a conditional Cauchy sequence in X. Since X is conditionally compact, (x_n) has a conditional cluster point $x \in X$, by Proposition 4.5.3. Due to Corollary 4.4.3, there exists a conditional subsequence (x_{n_m}) conditionally converging to x. Given $\varepsilon \in \mathbf{R}_{++}$, there exists $n_0 \in \mathbf{N}$ such that $d(x_n, x_m) < \varepsilon/2$ for every $n, m \geqslant n_0$. There exists also $m_0 \in \mathbf{N}$ such that $d(x, x_{n_k}) < \varepsilon/2$ for every $n_k \geqslant m_0$. By the conditional triangular inequality, it holds

$$d(x, x_n) \leqslant d(x, x_{n_k}) + d(x_{n_k}, x_n) < \varepsilon/2 + \varepsilon/2 = \varepsilon, \quad n, n_k \geqslant \max\{n_0, m_0\}.$$

Hence, $x_n \to x$, and thus X is conditionally complete.

(Step 2) We prove that *(ii)* implies *(iii)*. Let $(x_n)_{n\in\mathbf{N}}$ be a conditional sequence. Let

$$a := \vee\{b \in \mathcal{A} : \{bx_n : n \in b\mathbf{N}\} \text{ is conditionally finite}\}.$$

If $a = 1$, then (x_n) has a conditionally convergent subsequence. Otherwise, it suffices to find a conditionally converging subsequence on $a^c > 0$. Without loss of generality, suppose that $a = 0$. Since (X, d) is conditionally totally bounded, for every $N \in \mathbb{N}$ there exists $(y_i^N)_{1\leqslant i\leqslant k(N)}$ such that $X = \sqcup_{1\leqslant i\leqslant k(N)} B_{1/N}(y_i^N)$. For each $n \in \mathbf{N}$ there exists some $1 \leqslant i_0 \leqslant k(N)$ such that $x_n \in B_{1/N}(y_{i_0}^N)$ since the family $(B_{1/N}(y_i^N))_{1\leqslant i\leqslant k(N)}$ is conditional. Thus, $x_n \in B_1(y_{i_1^0}^1) \sqcap \ldots \sqcap B_{1/N}(y_{i_N^0}^N)$ for each $n \in \mathbf{N}$. Since

$$\begin{aligned}
(x_n)_{n\in\mathbf{N}} \subseteq X &= \bigsqcup_{1\leqslant i_1\leqslant k(1)} B_1(y_{i_1}^1) \sqcap \ldots \sqcap \bigsqcup_{1\leqslant i_N\leqslant k(N)} B_{\frac{1}{N}}(y_{i_N}^N) \\
&= \bigsqcup_{1\leqslant i_1\leqslant k(1)} \cdots \bigsqcup_{1\leqslant i_N\leqslant k(N)} B_1(y_{i_1}^1) \sqcap \ldots \sqcap B_{\frac{1}{N}}(y_{i_N}^N) \\
&= \sqcup_{1\leqslant i_1\leqslant k(1)} \cdots \sqcup_{1\leqslant i_N\leqslant k(N)} B_1(y_{i_1}^1) \sqcap \ldots \sqcap B_{\frac{1}{N}}(y_{i_N}^N),
\end{aligned}$$

there exist a conditional intersection $B_1(y_{i_1^0}^1) \sqcap \ldots \sqcap B_{1/N}(y_{i_N^0}^N)$ which contains conditionally countably infinite of the x_n, by Lemma 3.4.7. By induction, one finds a sequence (z_N) such that for each $N \in \mathbb{N}$ there exists some $n_0 \in \mathbb{N}$ and $d(z_M, z_L) \leqslant 1/N$ for all $M, L \geqslant n_0$. Define $z_{\sum a_j N_j} := \sum a_j z_{N_j}$ for every partition of unity (a_j) and $(N_j) \subseteq \mathbb{N}$. Then (z_n) is a conditional subsequence of (x_n) which is conditionally Cauchy. Since (X, d) is conditionally complete, (z_k) conditionally converges to some $z \in X$.

(Step 3) As for the implication *(iii)* to *(i)*, let (O^i) be a conditional open covering of X. By Lemma 5.5.5, there exists $\varepsilon \in \mathbf{R}_{++}$ such that for every $x \in X$ there exists an i and $B_\varepsilon(x) \sqsubseteq O^i$. We want to find a conditionally finite family $(x_j)_{1\leqslant j\leqslant n}$ such that

$X = \sqcup_{1\leqslant j\leqslant n} B_\varepsilon(x_j)$. Define

$$a = \vee\{b \in \mathcal{A} : bX = b \sqcup_{1\leqslant j\leqslant n} B_\varepsilon(x_j), (x_j)_{1\leqslant j\leqslant n} \text{ conditionally finite}\}.$$

By Assumption **(P)** and $\mathcal{A}$-stability, a is attained. If $a = 1$, we are done. Suppose, for the sake of contradiction, that $a^c > 0$. Assume, without loss of generality, that $a = 0$. Let $x_1 \in X$, $X_1 := \{x_1\}^\sqsubset$, and

$$b_1 := \vee\{b \in \mathcal{A} : d(bx_1, bx) \geqslant \varepsilon \text{ for some } x \in X_1\}.$$

If $b_1^c > 0$, then $b_1^c X$ is conditionally covered by $B_{b_1^c\varepsilon}(b_1^c x_1)$ which contradicts the maximality of a. By Assumption **(P)** and $\mathcal{A}$-stability, there exists $x_2 \in X_1$ such that $d(x_1, x_2) \geqslant \varepsilon$. Assume that we have found $x_1, x_2, \ldots, x_n$ such that $d(x_k, x_{k+1}) \geqslant \varepsilon$ for $k = 1, \ldots, n-1$. Let $X_n := cond(\{x_1, x_2, \ldots, x_n\})^\sqsubset$ and

$$b_n := \vee\{b \in \mathcal{A} : d(bx_k, bx) \geqslant \varepsilon \text{ for all } k = 1, \ldots, n \text{ and for some } x \in X_n\}.$$

If $b_n^c > 0$, then $b_n^c X$ is conditionally covered by $\sqcup_{k=1}^n B_{b_n^c\varepsilon}(b_n^c x_k)$ which contradicts the maximality of a. Hence, there exists $x_{n+1} \in X_n$ such that $d(x_n, x_{n+1}) \geqslant \varepsilon$. By induction we have found a sequence $(x_n)_{n\in\mathbb{N}}$ such that $d(x_n, x_m) \geqslant \varepsilon$ for all $n, m \in \mathbb{N}$ with $n \neq m$. Setting $x_n := \sum a_i x_{n_i}$ for $n = \sum a_i n_i \in \mathbf{N}$ defines a conditional sequence (x_n) such that $d(x_n, x_m) \geqslant \varepsilon$ for every $n, m \in \mathbf{N}$ with $n \sqcap m = \mathbf{0}$. Then (x_n) does not have a conditionally converging subsequence. However, this contradicts the assumptions. $\square$

Proposition 5.5.7. *Let (X, d) be a conditionally complete metric space and $Y \in \mathcal{S}(X)$. Then (Y, d) is conditionally complete if and only if Y is conditionally closed.*

Proof. Let Y be conditionally closed. Since every conditional Cauchy sequence (x_n) in Y is a conditional Cauchy sequence in X and since X is conditionally complete, (x_n) is conditionally convergent. By conditional closedness of Y, it follows that the conditional limit of (x_n) is in Y, by Proposition 4.4.6. Thus, Y is conditionally complete. Conversely, let Y be conditionally complete and (x_n) be conditionally convergent in Y. By the conditional triangular inequality, every conditionally convergent sequence is conditionally Cauchy. It follows that $\mathrm{Lim}(x_n) \in Y$. Thus, Y is conditionally closed, by Proposition 4.4.6. $\square$

We prove the conditional version of the Heine-Borel Theorem.

Theorem 5.5.8. *A conditional set $Y \in \mathcal{S}(\mathbf{R}^n)$ is conditionally closed and bounded if and only if it is conditionally compact.*

Proof. By Theorem 5.5.6, it suffices to show that Y is conditionally closed and bounded if and only if Y is conditionally totally bounded and complete. The equivalence between conditional closedness and completeness follows from Proposition 5.2.18 and 5.5.7. Let

Y be conditionally totally bounded and $N \in \mathbf{N}$. For every $N = \sum b_j n_j \in \mathbf{N}$ there exists a partition of unity $(a_i)_{i \in I}$ and for each $i \in I$ a finite family J_i such that

$$Y = \sum a_i \bigsqcup_{k \in J_i} B_{\frac{1}{N}}(x_k),$$

due to Lemma 3.4.7. Then Y is conditionally bounded by $m = \sum_{i,j} (a_i \wedge b_j) 2/n_j \, |J_i|$. Conversely, assume that Y is conditionally bounded. Let $\varepsilon \in \mathbf{R}_{++}$ and

$$a = \vee \{ b \in \mathcal{A} : bY = b \sqcup_{1 \leqslant j \leqslant n} B_\varepsilon(x_j), (x_j)_{1 \leqslant j \leqslant n} \text{ conditionally finite} \}.$$

If $a^c > 0$, then there exists a conditional sequence (x_n) in $a^c Y$ which exceeds every conditional bound, by *(Step 3)* of the proof of Theorem 5.5.6. □

6 The Category of Conditional Sets

The category of sheaves for a Grothendieck topology on a small category forms an elementary topos. It is well-known that the category of sheaves for the sup-topology on some $\mathcal{A} \in \mathscr{A}$ is a Boolean topos with a natural numbers object and satisfying the axiom of choice, which is, however, well-pointed if and only if $\mathcal{A}$ is trivial. We reconstruct this result in Section 6.1. In [LM92, Chapter I-VI] one finds an abstract construction and characterization of toposes of sheaves on an arbitrary (small) category. A Boolean algebra $\mathcal{A} \in \mathscr{A}$ is such a small category. We translate and apply the abstract results to the concrete case of some $\mathcal{A} \in \mathscr{A}$. In Section 6.2, we discuss the connection of conditional sets to sheaves. In Section 6.3, we build a category $\mathfrak{C}$ of conditional sets in accordance with the definitions in Chapter 3. We show that $\mathfrak{C}$ does not satisfy all properties of a Boolean topos with a natural numbers object and axiom of choice.

6.1 Sheaves on Boolean algebras

For the remainder of this section fix some non-degenerate $\mathcal{A} = (\mathcal{A}, \wedge, \vee, {}^c, 0, 1) \in \mathscr{A}$. The Boolean algebra $\mathcal{A}$ can be regarded as a category. The objects of $\mathcal{A}$ are the elements of the set $\mathcal{A}$. Its arrows consist of all ordered pairs (a, b) such that $a \leq b$. The domain of (a, b) is its first component a and its co-domain its second one b. The composition of two arrows (b, c) and (a, b) is defined by the transitivity of $\leq$ as $(b, c) \circ (a, b) = (a, c)$. The identity map of an object a is just (a, a). We denote an arrow in $\mathcal{A}$ by either (a, b), $a \leq b$, or $a \to b$, and by $\Delta = \Delta(\mathcal{A})$ the collection of arrows of $\mathcal{A}$ and $\Delta_a := \Delta(\mathcal{A}_a)$ for any $a \in \mathcal{A}$. If there is no risk of confusion, we identify an arrow (a, b) with its domain a. Note that every arrow $(a, b) \in \Delta$ is epic and monic, however, the identities (a, a) are the only isomorphisms.

For every object $a \in \mathcal{A}$ there exists exactly one arrow $(0, a)$ and $(a, 1)$. Thus, 0 is the initial and 1 the terminal object of $\mathcal{A}$. Given two arrows $a \to c$ and $b \to c$, their pullback is $a \wedge b$ together with the two projections $(a \wedge b, a)$ and $(a \wedge b, b)$. Indeed, for every other object d such that $(a, c) \circ (d, a) = (d, c) = (b, c) \circ (d, b)$ there exists exactly one arrow $d \leq a \wedge b$. In particular, the product of two objects a and b is given by $a \wedge b$. Similarly, one can show that $b \vee c$ is the pushout for any $a \to b$ and $a \to c$.

Definition 6.1.1 ([LM92]). A *presheaf* on $\mathcal{A}$ is a functor $X : \mathcal{A}^{\mathrm{op}} \to \mathbf{Sets}$. More precisely, X assigns to each object $a \in \mathcal{A}$ a set X_a and to each arrow $a \leq b$ a function $\gamma_a^b : X_b \to X_a$ satisfying

(i) $\gamma_a^a : X_a \to X_a$ is the identity for every $a \in \mathcal{A}$;

(ii) $\gamma_a^b \circ \gamma_b^c = \gamma_a^c$ for all $a \leq b \leq c$.

By $X = (X,\gamma) = ((X_a)_{a\in\mathcal{A}}, (\gamma^b_a)_{(a,b)\in\Delta})$ we denote a presheaf on $\mathcal{A}$. Let (X,γ) and (Y,δ) be two presheaves. A *map of presheaves* is a natural transformation $f : X \to Y$, that is a family of functions $f = (f_a)_{a\in\mathcal{A}}$ such that the diagram

$$\begin{tikzcd} X_b \arrow[r, "f_b"] \arrow[d, "\gamma^b_a"'] & Y_b \arrow[d, "\delta^b_a"] \\ X_a \arrow[r, "f_a"'] & Y_a \end{tikzcd}$$

commutes for every $(a,b) \in \Delta$. For each $a \in \mathcal{A}$, f_a is called a component of f.

Definition 6.1.2 ([LM92]). A family $S = (a_i)_{i\in I} = \{a_i \le a : i \in I\}$ of arrows in $\mathcal{A}$ is a *sieve* on a if $b \le a_i$ for some $i \in I$ implies $b \le a \in S$.

Examples 6.1.3. (i) The *maximal sieve* on a is the family of all arrows with codomain a, [LM92, p. 110].

(ii) Every partition $(a_i)_{i\in I} \in \mathcal{K}(a)$ generates a sieve on $a \in \mathcal{A}$. Indeed, the family $\{b \le a \in \Delta : b \le a_i$ for some $i \in I\}$ is a sieve on a containing (a_i).

(iii) For every sieve $S = \{a_i \le a : i \in I\}$ on a and every arrow (b,a) in $\mathcal{A}$, the family $b^*(S) := \{a_i \le b : a_i \le a, i \in I\}$ is a sieve on b. ♢

Definition 6.1.4 (Definition 1, III.2, [LM92]). A *Grothendieck topology* on $\mathcal{A}$ is a function J which assigns to each object $a \in \mathcal{A}$ a collection of sieves $J(a)$ such that

(i) the maximal sieve is in $J(a)$;

(ii) if $S \in J(a)$, then $b^*(S) \in J(b)$ for any arrow $b \le a$;

(iii) if S is any sieve on a such that $b^*(S) \in J(b)$ for all $b \le a$ in $\mathcal{A}$, then $S \in J(a)$.

The elements of $J(a)$ are called the *covers* of a, and the category $\mathcal{A}$ endowed with a Grothendieck topology J is called a *site*.

Example 6.1.5 (p. 115, [LM92]). The *sup-topology* assigns to each object $a \in \mathcal{A}$ the collection of all sieves S satisfying $\vee S = a$. Since $\vee_\emptyset = 0$, it holds $\emptyset \in J(0)$. ♢

Definition 6.1.6 (p. 121-122, [LM92]). Let (X,γ) be a presheaf, J be a topology, and $(a_i)_{i\in I}$ be a cover of some object $a \in \mathcal{A}$. A *matching family* for $(a_i)_{i\in I}$ of elements of X is a function which assigns to each $i \in I$ an element $x_i \in X_{a_i}$, in such a way that $\gamma^{a_i}_{a_j}(x_i) = x_j$ for all $j \in I$ with $a_j \le a_i$. An *amalgamation* of such a matching family is a single element $x \in X_a$ with

$$\gamma^a_{a_i}(x) = x_i, \quad \text{for all } i \in I. \tag{6.1.1}$$

A presheaf is a *sheaf* (for J) if every matching family for any cover of any object of $\mathcal{A}$ has a unique amalgamation.

The sheaves on some site $(\mathcal{A}, J)$ form a Grothendieck topos, where the maps are the maps of presheaves, [LM92, p. 127]. By $\mathbf{Sh} = \mathbf{Sh}(\mathcal{A}, J)$ we denote the topos of sheaves on $(\mathcal{A}, J)$.

Remark 6.1.7. If X is sheaf for J and $\emptyset \in J(a)$ for some $a \in \mathcal{A}$, then X_a is a singleton. Indeed, there exists exactly one matching family for the empty cover which is the empty family. By definition, there is an amalgamation $x \in X_a$ for this family which satisfies (6.1.1) vacuously. Since every $x \in X_a$ satisfies (6.1.1), uniqueness of amalgamations implies that X_a is a singleton, see also [LM92, p. 149]. ♦

Definition 6.1.8 (Definition 2, III.2, [LM92]). A *Grothendieck basis* on $\mathcal{A}$ is a function K which assigns to each object $a \in \mathcal{A}$ a collection of families of arrows with co-domain a such that

(i) $\{(a, a)\} \in K(a)$;

(ii) if $\{(b_i, a) : i \in I\} \in K(a)$, then for every arrow (c, a) the family of pullbacks $\{(b_i \wedge c, c) : i \in I\} \in K(c)$;

(iii) if $\{(b_i, a) : i \in I\} \in K(a)$, and if for each $i \in I$ one has a family $\{(c_{ij}, b_i) : j \in J_i\}$ in $K(b_i)$, then the family of composites $\{(c_{ij}, a) : i \in I, j \in J_i\}$ is in $K(a)$.

The elements of $K(a)$ are called *covers* of a and the category $\mathcal{A}$ endowed with a basis K is a *site*. A topology J is said to be *generated* by a basis K if for all $S \in J(a)$ there exists $R \in K(a)$ such that $R \subseteq S$ for all $a \in \mathcal{A}$.

By [LM92, III.4, Proposition 1], one can describe all sheaves for a topology J purely in terms of a basis K. Let (X, γ) be a presheaf and $(a_i)_{i \in I} \in K(a)$. A family $(x_i) \in \prod X_{a_i}$ is matching for (a_i) if $\gamma^{a_i}_{a_i \wedge a_j}(x_i) = \gamma^{a_j}_{a_i \wedge a_j}(x_j)$ for all $i, j \in I$. An amalgamation of (x_i) is a unique $x \in X_a$ with the property that $\gamma^a_{a_i}(x) = x_i$ for all $i \in I$. A presheaf X is a sheaf for J if and only if for any cover in the basis K, any matching family has a unique amalgamation.

Example 6.1.9. The function $\mathcal{K}$ which assigns to each object $a \in \mathcal{A}$ the collection of all of its partitions forms a Grothendieck basis, and we call $\mathcal{K}$ the *basis of partitions*. Note that $\emptyset \in \mathcal{K}(0)$. ♢

Proposition 6.1.10. *On every $\mathcal{A} \in \mathscr{A}$ the basis of partitions generates the sup-topology J.*

Proof. By Assumption **(P)**, for every $(a_i) \in J(a)$ for some $a \in \mathcal{A}$ there exists $(b_j) \in \mathcal{K}(a)$ such that for all $j \in J$ there is $i \in I$ and $b_j \leq a_{i_j}$. Since (a_i) is a sieve, it follows that $\{b_j : j \in J\} \subseteq \{a_i : i \in I\}$. □

Hence, the category of sheaves on the site $(\mathcal{A}, \mathcal{K})$ is equivalent to the category of all sheaves on $(\mathcal{A}, J)$ where J is the sup-topology. $\mathbf{Sh}(\mathcal{A}, \mathcal{K})$ is an elementary topos, by [LM92, III.4, Definition 3] and [LM92, III.7, Corollary 4]. In the remaining part of this section, we reconstruct the elements of $\mathbf{Sh}$ which is a Boolean topos with a natural numbers objects and satisfying the axiom of choice, which, however, is two-valued if and only if $\mathcal{A}$ is the trivial algebra.

Remark 6.1.11. Let (X,γ) be a sheaf in **Sh** and $(a_i)_{i\in I}\in\mathcal{K}(a)$ for some $a\in\mathcal{A}$. Then every family $(x_i)\in\prod X_{a_i}$ is matching for (a_i). Indeed,

$$\gamma^{a_i}_{a_i\wedge a_j}(x_i)=\gamma^{a_i}_0(x_i)=x_0=\gamma^{a_j}_0(x_j)=\gamma^{a_j}_{a_i\wedge a_j}(x_j),\quad \text{for all } i,j\in I,$$

since $X_0=\{x_0\}$ is a singleton. For $(a_i)\in\mathcal{K}(a)$ and $(x_i)\in\prod X_{a_i}$, we denote by $\sum x_i$ the amalgamation of the family (x_i). If I is finite, say $I=\{1,\dots,n\}$, we write $x_1+\ldots+x_n$.♦

Limits and co-limits in **Sh** are computed pointwise, [LM92, Sections I.2 and III.6]. Terminal object of **Sh** is the constant sheaf $\mathbf{1}$ where $\mathbf{1}_a:=\{*\}$ for each $a\in\mathcal{A}$, and the initial object is the sheaf $\mathbf{0}$ given by $\mathbf{0}_a:=\emptyset$ for $a>0$ and $\mathbf{0}_0:=\{*\}$, [LM92, p. 149]. Let $f:X\to Z$ and $g:Y\to Z$ be two natural transformations, and $X_a\times_{Z_a}Y_a$ together with the projections π^X_a and π^Y_a be the pullback of the functions $f_a:X_a\to Z_a$ and $g_a:Y_a\to Z_a$ in the category **Sets**. Then the pullback of $f:X\to Z$ and $g:Y\to Z$ is given by $(X\times_Z Y)_a=X_a\times_{Z_a}Y_a$ for each $a\in\mathcal{A}$ together with the projection maps $\pi^X=(\pi^X_a)_{a\in\mathcal{A}}$ and $\pi^Y=(\pi^Y_a)_{a\in\mathcal{A}}$. The product $X\times Y$ of two sheaves X and Y is defined by the pullback of $X\to\mathbf{1}$ and $Y\to\mathbf{1}$. Let $f:X\to Y$ and $g:X\to Z$ be two natural transformations. The pushout of $f:X\to Y$ and $g:X\to Z$ is given by

$$\Big(X\coprod_Z Y\Big)_a=X_a\coprod_{Z_a}Y_a,\quad a\in\mathcal{A},$$

together with the respective pointwise embeddings. Monics and epics in **Sh** are also taken pointwise, [LM92, p. 37]. A natural transformation $(f_a)_{a\in\mathcal{A}}$ all components of which are injections is a monic and an isomorphism is a natural transformation all components of which are bijections.

Recall that the collection of subobjects $\mathrm{Sub}(X)$ of some sheaf (X,γ) is the set of all equivalence classes of monics into X. The subobject classifier of **Sh** is the sheaf Ω given on objects by $\Omega_a=\mathcal{A}_a$ and on arrows $a\le b$ by $c\mapsto c\wedge a$ for $c\in\mathcal{A}_a$, [LM92, p. 159-160]. The monic **true**: $\mathbf{1}\to\Omega$ is defined on components by $\mathbf{true}_a(*)=a$ for each $a\in\mathcal{A}$. Let $m:Y\to X$ be a subobject of some sheaf X. The characteristic map of m is the natural transformation $\psi=\psi_Y:X\to\Omega$, given by its components as

$$\psi_a(x):=\vee\{b\le a:\gamma^a_b(x)\in Y_b\},\quad x\in X_a.$$

It is the unique arrow for which the following square is a pullback:

$$\begin{tikzcd} Y \arrow[r] \arrow[d, "m"'] & \mathbf{1} \arrow[d, "\mathbf{true}"] \\ X \arrow[r, dashed, "\phi"'] & \Omega. \end{tikzcd}$$

Every subobject $m:Y\to X$ of some sheaf X in **Sh** corresponds uniquely to a subsheaf of X.

Definition 6.1.12 ([LM92]). A sheaf (Y,δ) is called a *subsheaf* of X if

(i) $Y_a \subseteq X_a$ for all $a \in \mathcal{A}$;

(ii) δ_a^b is the restriction of γ_a^b to Y_b for all $(a, b) \in \Delta$;

(iii) for each $a \in \mathcal{A}$, every cover (a_i) of a, and all $x \in X_a$ for which holds $\gamma_{a_i}^a(x) \in Y_{a_i}$ for every i, it already holds that $x \in Y_a$.

Every subsheaf Y of X defines a monic into X by inclusion. Thus, every subsheaf is a subobject. Conversely, let $m : Y \rightarrowtail X$ be given by the family of injections $m_a : Y_a \to X_a$. Set $Z_a := m_a(Y_a)$ and η_a^b be the restriction of γ_a^b on Z_a. In this way Z is a subsheaf of X whose inclusion map is equivalent to m as subobjects, see also [LM92, p. 37] and [LM92, III.7, Lemma 2].

For every sheaf X, the natural partial order on $\mathrm{Sub}(X)$ is expressed on subsheaves by

$$Y \leq Z \quad \text{if and only if} \quad Y_a \subseteq Z_a, \quad \text{for all } a \in \mathcal{A}.$$

The least subsheaf is the initial object $\mathbf{0}$ and the greatest one is X itself. The infimum of a family (Y^i) of subsheaves of X is defined as their pointwise intersection

$$(\wedge Y^i)_a := \cap Y_a^i, \quad a \in \mathcal{A},$$

and the supremum of (Y^i) is defined by

$$\vee Y^i := \wedge\{Z : Y^i \leq Z \text{ for all } i\}.$$

Equivalently, one characterizes the supremum pointwise as the set of those elements of X_a for which holds

$$x \in (\vee Y^i)_a \quad \text{if and only if} \quad \{b \in \mathcal{A} : b \leq a \text{ and } \gamma_b^a(x) \in Y_b^i, \text{ for some } i\} \in \mathcal{K}(a). \tag{6.1.2}$$

The pseudo-complement of a subsheaf Y in $\mathrm{Sub}(X)$ is given by

$$\neg Y := \vee\{Z \,:\, Z \wedge Y = \mathbf{0}\}. \tag{6.1.3}$$

Equivalently, $x \in X_a$ is an element of $\neg Y_a$ if and only if $\gamma_a^b(x) \in E_a$ for some $a \leq b$ implies $b = 0$. The aforementioned definitions of the lattice operations in $\mathrm{Sub}(X)$ is based on [LM92, p. 145-146]. For every sheaf X in **Sh**, the lattice of subsheaves $\mathrm{Sub}(X)$ is a complete Heyting algebra, by [LM92, III.8, Proposition 1]. It is known that $\mathrm{Sub}(X)$ is actually a complete Boolean algebra. We give a prove, for the sake of completness.

Proposition 6.1.13. *The Heyting algebra Sub(X) is Boolean for every sheaf X.*

Proof. Recall that $\mathrm{Sub}(X)$ is Boolean if and only if $Y \vee \neg Y = X$ for every Y in $\mathrm{Sub}(X)$, by [LM92, I.8, Proposition 4]. Clearly, $Y \vee \neg Y \leq X$. For the reverse inclusion, let $x \in X_a$ for some object $a \in \mathcal{A}$. Without loss of generality, assume $a > 0$ and let

$$M := \{b \leq a \,:\, \gamma_b^a(x) \in Y_b\}.$$

Then M is a sieve on a. Due to Assumption **(P)**, there exists a partition R of $\vee M$ such that for all $b \in R$ there is $c \in M$ with $b \leq c$ which yields $b \in M$. Let $b_* := \vee M$. The amalgamation of the family $(\gamma_b^a(x))_{b\in R}$ is $\gamma_{b_*}^a(x) \in Y_{b_*}$. Thus, $M = \{b \in \mathcal{A} : b \leq b_*\}$. If $b_* = a$, then $x \in Y_a$, and we are done. Otherwise, assume $b_*^c \wedge a > 0$. Then it suffices to show that $\vee N = a \wedge b_*^c$, where $N := \{b \leq a : \gamma_b^a(x) \in \neg Y_b\}$, since in this case x is represented by $\gamma_{b_*}^a(x) + \gamma_{a\wedge b_*^c}^a(x) \in (Y \vee \neg Y)_a$, due to (6.1.2). Note that $\vee N$ is the greatest element of N, as shown before for M. For the sake of contradiction, assume $c := (\vee N)^c \wedge b_*^c \wedge a > 0$. Due to the maximality of b_*, if $\gamma_d^a(x) \in Y_d$ for some $d \leq c$, then $d = 0$. This implies by the aforementioned characterization of $\neg Y$ that $\gamma_c^a(x) \in Y_c$ which is a contradiction. Hence, $c = 0$, and thus $\vee N = a \wedge b_*^c$. □

Definition 6.1.14 ([LM92]). Let (X, γ) be a sheaf for some topology J on $\mathcal{A}$ and J_a be the restriction of J on the relative algebra $\mathcal{A}_a$ defined by $J_a(b) := J(b)$ for every $b \in \mathcal{A}_a$. The sheaf

$$aX := \Big((X_b)_{b\in\mathcal{A}_a}, (\gamma_a^b)_{(a,b)\in\Delta(\mathcal{A}_a)}\Big)$$

on the site $(\mathcal{A}_a, J_a)$ is the *restriction* of X on a.

Note that if Y is a subsheaf of X, then aY is a subsheaf of aX for all $a \in \mathcal{A}$.

Definition 6.1.15 (p. 138, [LM92]). For two sheaves X and Y in **Sh**, their *exponential* is the sheaf $Y^X = (Y^X, \alpha)$, given on objects by $Y_a^X := \mathrm{Hom}(aX, aY)$ for each $a \in \mathcal{A}$, and on every arrow $a \leq b \in \Delta$ by $\alpha_a^b : Y_b^X \to Y_a^X$ defined by $(f_c)_{c\in\mathcal{A}_b} \mapsto (f_c)_{c\in\mathcal{A}_a}$.

The power object of some sheaf X in **Sh** is the exponential Ω^X together with the natural transformation $\varepsilon^X : X \times \Omega^X \to \Omega$, defined by its components as

$$X_a \times \mathrm{Hom}(aX, a\Omega) \ni (x, \psi^a) \longmapsto \varepsilon_a^X(x, \psi^a) := \psi_a^a(x),$$

see [LM92, Section IV.2].

The natural numbers object of **Sh** is the sheafification of the constant presheaf N given by $N_a := \mathbb{N}$ for all $a \in \mathcal{A}$, [LM92, p. 270]. We reconstruct the sheafification of N. For every $a \in \mathcal{A}$ and $(a_i) \in \mathcal{K}(a)$, define $M(a_i) := \{(x_i) : x_i \in \mathbb{N}, i \in I\}$. Denote by M_a the union of $M(a_i)$ over all $(a_i) \in \mathcal{K}(a)$ for each $a \in \mathcal{A}$. Let (x_i) and (y_j) be in M_a for some (a_i) and (b_j) in $\mathcal{K}(a)$. Then (x_i) is equivalent to (y_j) if $\vee\{a_i : x_i = n\} = \vee\{b_j : y_j = n\}$ for all $n \in \mathbb{N}$. By $\mathbf{N}_a$ denote the set of equivalence classes of M_a. For every arrow $a \leq b$, the function $\delta_a^b : \mathbf{N}_b \to \mathbf{N}_a$ is given by $\delta_a^b((x_i)) := (x_i)$, where the partition on the left-hand side is (a_i) and on the right-hand side $(a \wedge a_i)$. Then $\mathbf{N} = ((\mathbf{N}_a)_{a\in\mathcal{A}}, (\delta_a^b)_{(a,b)\in\Delta})$.

We verify that **Sh** is generated by the subsheaves of $\mathbf{1}$. That implies that for every pair of natural transformations $f : X \to Y$ and $g : X \to Y$ in **Sh** with $f \neq g$ there exists a subobject Z in $\mathrm{Sub}(\mathbf{1})$ and a natural transformation $u : Z \to X$ such that $f \circ u \neq g \circ u$. Let $f : X \to Y$ and $g : X \to Y$ be in **Sh** such that $f \neq g$. Then there exist $c \in \mathcal{A}$ and $x_c \in X_c$ such that $f_c(x_c) \neq g_c(x_c)$. Let $\theta : \mathbf{1} \to \Omega$ be given by $\theta_b = c \wedge b$ for all $b \in \mathcal{A}$ which is a monic. Hence, we can associate a subsheaf Z of $\mathbf{1}$ to θ which is given on objects by $Z_b = 1$ if $b \leq c$ and $Z_b = \emptyset$ otherwise. Define the natural transformation

$u : Z \to X$ on components by $u_b = \gamma_b^c(x_c)$ if $b \le c$, and $u_b = \emptyset$ otherwise. It holds $(f \circ u)_c = f(x_c) \ne g(x_c) = (g \circ u)_c$, and thus $f \circ u \ne g \circ u$.

Since **Sh** is Boolean and is generated by subobjects of **1**, **Sh** satisfies the axiom of choice, by [LM92, VI.2, Proposition 8]. Finally, to verify that $\mathbf{Sh}(\mathcal{A}, \mathcal{K})$ is well-pointed if and only if $\mathcal{A} = \{0, 1\}$, it suffices to show that **Sh** is two-valued if and only if $\mathcal{A}$ is the trivial algebra, by [LM92, VI.2, Proposition 7]. Recall that a topos is two-valued if $\mathrm{Sub}(\mathbf{1}) = \{\mathbf{0}, \mathbf{1}\}$. It follows immediately, that $\mathrm{Sub}(\mathbf{1})$ consist of **0** and **1** if and only if $\mathcal{A}$ is trivial. In this case, **Sh** is equivalent to **Sets**.

6.2 Sheaves and conditional sets

Recall the definition of a conditional set.

Definition 6.2.1. Let $\mathcal{A} \in \mathscr{A}$, $(X_a)_{a\in\mathcal{A}}$ be a family of sets, and $(\gamma_a)_{a\in\mathcal{A}}$ be a family of surjective functions $\gamma_a : X_1 \to X_a$. The structure

$$X := (X_a, \gamma_a)_{a\in\mathcal{A}}$$

is a *conditional set* if and only if

(i) X_0 is a singleton;

(ii) *(identity)* γ_1 is the identity;

(iii) *(consistency)* $\gamma_a(x) = \gamma_a(y)$, whenever $\gamma_b(x) = \gamma_b(y)$, $x, y \in X_1$, and $(a, b) \in \Delta$;

(iv) *($\mathcal{A}$-stability)* for every partition of unity $(a_i)_{i\in I}$ and for every $(x_i)_{i\in I} \in \prod_{i\in I} X_{a_i}$ there exists a unique $x \in X_1$ such that $\gamma_{a_i}(x) = x_i$ for all $i \in I$.

Proposition 3.1.4 says that every conditional set generates a sheaf on the site $(\mathcal{A}, \mathcal{K})$ which is given on arrows by surjective functions. Hence, the collection of all conditional sets on some fixed non-degenerate $\mathcal{A} \in \mathscr{A}$ can be identified with the subcategory of all sheaves which are given on arrows by surjections and natural transformations between them. Denote by $\mathfrak{S}$ that subcategory.

The assumptions of surjectivity and **(P)** simplify the structure of a sheaf in the following aspects:

- The sup-topology is generated by the basis of partitions, due to Proposition 6.1.10.
- For every sheaf X on the site $(\mathcal{A}, \mathcal{K})$, each family for some cover is matching, due to Remark 6.1.11.
- For any sheaf (X, γ) in $\mathfrak{S}$, the family of functions $(\gamma_a^1)_{a\in\mathcal{A}}$ already determines the whole family $(\gamma_a^b)_{(a,b)\in\Delta}$, due to Proposition 3.1.4.
- For every sheaf X in $\mathfrak{S}$, it holds $X_a \ne \emptyset$ for all $a \in \mathcal{A}$, due to surjectivity and $\emptyset \in \mathcal{K}(0)$.

- Properties of sheaves in $\mathfrak{S}$ can be expressed and proved in terms of X_1, as shown throughout this thesis.

There are the following differences between conditional set theory and toposes of sheaves. The collection of conditional sets $\mathcal{C}$ on some fixed $\mathcal{A} \in \mathscr{A}$ can be associated to the subcategory $\mathfrak{S}$ of $\mathbf{Sh}(\mathcal{A}, \mathcal{K})$. On $\mathfrak{S}$ the conditional inclusion coincides with the natural partial order on $\mathrm{Sub}(X)$ for some fixed $X \in \mathcal{C}$. More precisely, it holds $\mathcal{S}(X) \subseteq \mathrm{Sub}(X)$. However, in $\mathrm{Sub}(X)$ there may exist sheaves whose arrows are not surjective. For instance, the pointwise intersection of two $Y, Z \in \mathcal{S}(X)$ whose conditional intersection lives on some condition a strictly smaller than 1. In this case, the intersection of Y_b and Z_b is empty for all $b \in \mathcal{A}$ with $b \wedge a^c > 0$. The pointwise intersection is a sheaf, as shown in the previous section, however, it is no more a conditional set. Pointwise intersection and conditional intersection of Z and Y in $\mathcal{S}(X)$ coincide if and only if $Z \sqcap Y \in \mathcal{S}(X)$. Similarly, the conditional complement of some $Y \in \mathcal{S}(X)$ may fall out of $\mathfrak{S}$. There is no relation between $\mathcal{P}(X)$ and $\mathrm{Sub}(X)$. The conditional inclusion relation is defined on the class of all conditional sets. For a given conditional set X on some $\mathcal{A}$, one has to consider at least the class of all conditional sets on all relative algebras of $\mathcal{A}$, in order to introduce the conditional inclusion. The class $\mathfrak{S}$ is not sufficient. It holds the following proposition.

Proposition 6.2.2. *Let $\mathcal{A} \in \mathscr{A}$ and X be a conditional set on $\mathcal{A}_a$ for some $a \in \mathcal{A}$. Then $X = aY$ for some conditional set Y on $\mathcal{A}$.*

Proof. Let $X = (X_b, \gamma_b)_{b \in \mathcal{A}_a}$ and define $Y = (Y_b, \delta_b)_{b \in \mathcal{A}}$ by $Y_b := X_{b \wedge a}$ and $\delta_b := \gamma_{b \wedge a}$ for all $b \in \mathcal{A}$. By construction, $aY = X$. □

Thus, one has to consider the class of all restrictions of elements of $\mathfrak{S}$. This motivates the construction of a category of conditional sets in the following section.

6.3 A category of conditional sets

In this section, our goal is to construct a suitable category for conditional sets in which the basic operations are expressed in terms of conditional functions. Fix some non-degenerate $\mathcal{A} \in \mathscr{A}$. Recall that $\mathfrak{S}$ denotes the subcategory of $\mathbf{Sh}$ consisting of sheaves whose arrows are given by surjections. Let $\mathcal{A} \times \mathfrak{S}$ be the product category of $\mathcal{A}$ and $\mathfrak{S}$. Recall that the objects of $\mathcal{A} \times \mathfrak{S}$ are pairs (a, X) where a is an object of $\mathcal{A}$ and X is a sheaf in $\mathfrak{S}$. An arrow in $\mathcal{A} \times \mathfrak{S}$ is a pair $((a, b), f)$, where $(a, b) \in \Delta$ and f is a natural transformation. Thus, the set of homomorphisms of two objects (a, X) and (b, Y) is given by

$$\mathrm{Hom}((a, X), (b, Y)) = \begin{cases} \{(a, f) : f \in \mathrm{Hom}(X, Y)\}, & \text{if } a \leq b, \\ \emptyset, & \text{otherwise.} \end{cases}$$

Define on $\mathcal{A} \times \mathfrak{S}$ a congruence R, as follows. If $\mathrm{Hom}((a,X),(b,Y))$ is empty, consider on it the empty relation, and otherwise define

$$(a,f) \sim (a,g) \quad \text{if and only if} \quad f_c = g_c, \quad \text{for all } c \leq a.$$

It is easy to see, that R defines a congruence on $\mathcal{A} \times \mathfrak{S}$.

Definition 6.3.1. The quotient category $\mathfrak{C} = \mathfrak{C}(\mathcal{A}) := \mathcal{A} \times \mathfrak{S}/_R$ is called the *category of conditional sets* with respect to $\mathcal{A}$. Its objects are called *conditional sets* and its arrows *conditional functions.*

Lemma 6.3.2. *Let $(X,\gamma),(Y,\delta) \in \mathfrak{S}$, $a \leq b$, and $h : aX \to aY$ be a natural transformation. Then h extends uniquely to a conditional function $(a,f) : (a,X) \to (b,Y)$.*

Proof. Let $g : X \to Y$ be a natural transformation, chosen arbitrarily. For every $c \in \mathcal{A}$ and every $x \in X_c$, define

$$f_c(x) = h_{c\wedge a}(\gamma^c_{c\wedge a}(x)) + g_{c\wedge a^c}(\gamma^c_{c\wedge a^c}(x))$$

Due to the uniqueness of amalgamations, f_c is well-defined. We show that $(f_c)_{c\in\mathcal{A}}$ is a natural transformation from X into Y. Let $c' \leq c$ and $x \in X_c$. Define $x_a = \gamma^c_{a\wedge c}(x)$ and $x_{a^c} = \gamma^c_{a^c\wedge c}(x)$. Since h and g are natural transformations, it holds

$$h_{a\wedge c'}(\gamma^{a\wedge c}_{a\wedge c'}(x_a)) = \delta^{a\wedge c}_{a\wedge c'}(h_{a\wedge c}(x_a)) \quad \text{and} \quad g_{a^c\wedge c'}(\gamma^{a^c\wedge c}_{a^c\wedge c'}(x_{a^c})) = \delta^{a^c\wedge c}_{a^c\wedge c'}(g_{a^c\wedge c}(x_{a^c})).$$

This together with the uniqueness of amalgamations implies

$$\begin{aligned}\delta^c_{c'}(f_c(x)) &= \delta^{a\wedge c}_{a\wedge c'}(h_{a\wedge c}(x_a)) + \delta^{a^c\wedge c}_{a^c\wedge c'}(g_{a^c\wedge c}(x_{a^c}))\\ &= h_{a\wedge c'}(\gamma^{a\wedge c}_{a\wedge c'}(x_a)) + g_{a^c\wedge c'}(\gamma^{a^c\wedge c}_{a^c\wedge c'}(x_{a^c}))\\ &= f_{c'}(\gamma^c_{c'}(x)).\end{aligned}$$

Hence, $f : X \to Y$ is a natural transformation, and thus (a,f) is a conditional function from (a,X) into (b,Y). Let $g' : X \to Y$ be another natural transformation and construct f' analogously to f. Then $(a,f) \sim (a,f')$ since $f_c = f'_c$ for all $c \leq a$. □

Lemma 6.3.3. *An arrow $(a \leq b, m) : (a,X) \to (b,Y)$ is a monic in $\mathfrak{C}$ if and only if m_c is an injection for all $c \leq a$.*

Proof. We show that the condition is sufficient. Let m_c be injective for all $c \leq a$, and let

$$(c \leq a, h) : (c,Z) \to (a,X), \quad (c \leq a, g) : (c,Z) \to (a,X),$$

be such that

$$(a \leq b, m) \circ (c \leq a, h) \sim (a \leq b, m) \circ (c \leq a, g).$$

Hence, $(m \circ h)_d = (m \circ g)_d$ for all $d \leq c$. Since m_d is injective for all $d \leq c \leq a$, it holds that $h_d = g_d$ for all $d \leq c$. Thus, $(c \leq a, h) \sim (c \leq a, g)$.

To prove necessity, let $(X, \gamma), (Y, \delta) \in \mathfrak{S}$, $a \leq b$, $(a \leq b, m) : (a, X) \to (b, Y)$ be a monic. Assume, without loss of generality, that $a > 0$ (the injectivity of $m_0 = \mathrm{id}_{\{*\}}$ is trivially true). Suppose, for the sake of contradiction, that there exists some $0 < c \leq a$ and two elements x and y in X_c with $x \neq y$ such that $m_c(x) = m_c(y)$. Let $h : \mathbf{1} \to X$ be the global element generated by x. Recall that h is given by

$$h_d = \gamma^c_{c \wedge d}(x) + \gamma^{c^c}_{c^c \wedge d}(z_{c^c}), \quad d \in \mathcal{A},$$

where $z : \mathbf{1} \to X$ is some arbitrary global element. Define another global element $g : \mathbf{1} \to X$ by

$$g_d = \gamma^c_{c \wedge d}(y) + \gamma^{c^c}_{c^c \wedge d}(z_{c^c}), \quad d \in \mathcal{A}.$$

We show that $(a \leq b, m) \circ (a \leq a, h) \sim (a \leq b, m) \circ (a \leq a, g)$. To this end, let $d \leq a$. Then it holds

$$\begin{aligned}
(m \circ h)_d &= m_d(\gamma^c_{c \wedge d}(x) + \gamma^{c^c}_{c^c \wedge d}(z_{c^c})) \\
&= m_{c \wedge d}(\gamma^c_{c \wedge d}(x)) + m_{c^c \wedge d}(\gamma^{c^c}_{c^c \wedge d}(z_{c^c})) \\
&= \delta^c_{c \wedge d}(m_c(x)) + \delta^{c^c}_{c^c \wedge d}(m_{c^c}(z_{c^c})) \\
&= \delta^c_{c \wedge d}(m_c(y)) + \delta^{c^c}_{c^c \wedge d}(m_{c^c}(z_{c^c})) \\
&= m_{c \wedge d}(\gamma^c_{c \wedge d}(y)) + m_{c^c \wedge d}(\gamma^{c^c}_{c^c \wedge d}(z_{c^c})) \\
&= m_d(\gamma^c_{c \wedge d}(y) + \gamma^{c^c}_{c^c \wedge d}(z_{c^c})) \\
&= (m \circ g)_d.
\end{aligned}$$

Since $(a \leq b, m)$ is a monic, it holds $g = h$. In particular, $g_c = h_c$ which is a contradiction. □

Definition 6.3.4. A *conditional subset* of (a, X) is a conditional set (b, Y) such that $b \leq a$ and bY is a subsheaf of bX.

Lemma 6.3.5. *Every equivalence class of monics describes a conditional subset of X and every conditional subset of X is an equivalence class of monics into X.*

Proof. Let (b, Y) be a conditional subset of (a, X). Let $n_c : Y_c \hookrightarrow X_c$, $c \leq b$, be the inclusion map stemming from the fact that bY is a subsheaf of bX. Then $(b \leq a, m)$ is a monic from (b, Y) into (a, X) where m is the extension of n. Conversely, every monic $(b \leq a, m) : (b, Y) \to (a, X)$ describes a subsheaf bY of bX (see the discussion in the previous section), and thus a conditional subset. Let $(b \leq a, m) : (b, Y) \rightarrowtail (a, X)$ and $(c \leq a, n) : (c, Z) \rightarrowtail (a, X)$ be two equivalent monics, that is $(b, Y) \simeq (c, Z)$. Since the only isomorphisms in $\mathcal{A}$ are the identities, $b = c$, and since there exists a bijection from Y_d to Z_d for all $d \leq b$, it holds $bY \simeq bZ$. □

By the previous lemma, we can identify the subobjects of a conditional set with the collection of all of its conditional subsets. The natural partial order on subobjects of some conditional set (a, X) then reads as follows:

$$(c, Z) \leq (b, Y) \quad \text{if and only if} \quad cZ \text{ is a subsheaf of } cY.$$

This relation is the conditional inclusion relation on $\mathrm{Sub}(a, X)$. By Theorem 3.2.11, for every conditional set (a, X), $\mathrm{Sub}(a, X)$ is a complete Boolean algebra.

Proposition 6.3.6. *The category of conditional sets has all finite limits and colimits.*

Proof. The terminal object of $\mathfrak{C}$ is $(1, \mathbf{1})$ up to isomorphisms where $\mathbf{1}$ is the terminal object of **Sh**. Indeed, the arrow side of $\mathbf{1}$ consists of surjections, and for any object (a, X) in $\mathfrak{C}$ there exists only one conditional function from (a, X) to $(1, \mathbf{1})$ which is $(a \leq 1, X \to \mathbf{1})$ where $X \to \mathbf{1}$ is the unique map from X to $\mathbf{1}$. Analogously, one can show that $(0, \mathbf{0})$ is the initial object of $\mathfrak{C}$.

Let $(a, f) : (a, X) \to (c, Z)$ and $(b, g) : (b, Y) \to (c, Z)$ be given. Then $(a \wedge b, X \times_Z Y)$ together with the projection maps $(a \wedge b, \pi^X)$ and $(a \wedge b, \pi^Y)$ is the pullback where $X \times_Z Y$ together with projection maps π^X and π^Y is the pullback of $f : X \to Z$ and $g : Y \to Z$ in **Sh**. We show that the diagram

$$\begin{tikzcd} (a \wedge b, X \times_Z Y) \arrow[r] \arrow[d] & (a, X) \arrow[d] \\ (b, Y) \arrow[r] & (c, Z) \end{tikzcd} \tag{6.3.1}$$

is universal. Let $(a \wedge b, \tilde{\pi}^X) \sim (a \wedge b, \pi^X)$, $(a, \tilde{f}) \sim (a, f)$, $(a \wedge b, \tilde{\pi}^Y) \sim (a \wedge b, \pi^Y)$, and $(b, g) \sim (b, \tilde{g})$. Then

$$(a, \tilde{f}) \circ (a \wedge b, \tilde{\pi}^X) = (a \wedge b, \tilde{f} \circ \tilde{\pi}^X) \sim (a \wedge b, \tilde{g} \circ \tilde{\pi}^Y) = (b, \tilde{g}) \circ (a \wedge b, \tilde{\pi}^Y),$$

if and only if $\tilde{f}_c \circ \tilde{\pi}^X_c = \tilde{g}_c \circ \tilde{\pi}^Y_c$ for all $c \leq a$ which is true, since $f_c \circ \pi^X_x = g_c \circ \pi^Y_c$ for all $c \leq a$, $(a, \tilde{f}) \sim (a, f)$ and $(b, g) \sim (b, \tilde{g})$. The diagram (6.3.1) is universal since $a \wedge b$ is the pullback of a and b, and $X \times_Z Y$ is the pullback of $f : X \to Z$ and $f : Y \to Z$. Analogously, one can show that $\mathfrak{C}$ has all pushouts. By [Lan98, Chapter III and V], $\mathfrak{C}$ has all finite limits and colimits. □

The conditional product of two conditional sets (a, X) and (b, Y) is the pullback of $(a, X) \to (1, \mathbf{1})$ and $(b, Y) \to (1, \mathbf{1})$ and coincides with the conditional product introduced in Section 3.3.

Remark 6.3.7. Observe that there does not always exist a conditional function from the terminal object $(1, \mathbf{1})$ to a conditional set (a, X). It exists only in the case that $a = 1$, otherwise the set of conditional functions is empty. However, given a conditional set (a, X) where $a < 1$, a *conditional global element* of (a, X) is a conditional function $(b, \mathbf{1}) \to (a, X)$ where $b \leq a$. ♦

The conditional set $(1, \mathbf{N})$ is the natural numbers object of $\mathfrak{C}$, where $\mathbf{N}$ is the natural numbers object of $\mathbf{Sh}$, since the arrow part of $\mathbf{N}$ consists of surjective functions. Moreover, it holds

$$\mathbf{1} \xrightarrow{0} \mathbf{N} \xrightarrow{s} \mathbf{N} \tag{6.3.2}$$

such that for any sheaf $X \in \mathfrak{S}$ with arrows x and f, as below, there is a unique arrow h such that the diagram

$$\begin{tikzcd}
\mathbf{1} \arrow[r, "0"] \arrow[d, equal] & \mathbf{N} \arrow[r, "s"] \arrow[d, dashed, "h"] & \mathbf{N} \arrow[d, dashed, "h"] \\
\mathbf{1} \arrow[r, "x"'] & X \arrow[r, "f"'] & X
\end{tikzcd} \tag{6.3.3}$$

commutes. Since conditional functions and natural transformations coincide on 1 and replacing in (6.3.2) $\mathbf{1}$ by $(1, \mathbf{1})$ and $\mathbf{N}$ by $(1, \mathbf{N})$, the universality property of $(1, \mathbf{N})$ is implied by (6.3.3).

We show that $\mathfrak{C}$ is generated by the subobjects of $(1, \mathbf{1})$. Let $(a, f) \neq (a, g)$ be two conditional functions from (a, X) into (b, Y). Since $(a, f) \neq (a, g)$, there exists $c \leq a$ and $x_c \in X_c$ such that $f_c(x_c) \neq g_c(x_c)$. According to the construction at the end of Section 6.1, let $u : aX \to aY$ be such that $af \circ au \neq ag \circ au$, where by af we denote $f : X \to Y$ restricted to aX. Then $f \circ v \neq g \circ v$, where $v : (a, X) \to (b, Y)$ is the extension of u. Since $\mathbf{Sh}$ is two-valued if and only if $\mathcal{A}$ is trivial, $\mathfrak{C}$ is two-valued if and only if $\mathcal{A}$ is trivial.

The only possible candidate for the subobject classifier in $\mathfrak{C}$ is $(1, \Omega)$ together with the monic $(1, \mathbf{true}) : (1, \mathbf{1}) \rightarrowtail (1, \Omega)$ where Ω and $\mathbf{true} : \mathbf{1} \to \Omega$ form the subobject classifier of $\mathbf{Sh}$. Let $(a, m) : (a, X) \rightarrowtail (b, Y)$ be a monic in $\mathfrak{C}$. Then aX is a subsheaf of aY. Let $\hat{\phi} : aY \to a\Omega$ be the characteristic map of aY and $(b, \phi) : (b, Y) \to (1, \Omega)$ be the extension of $\hat{\phi}$. Then the diagram

$$\begin{tikzcd}
(a, X) \arrow[r] \arrow[d] & (1, \mathbf{1}) \arrow[d] \\
(b, Y) \arrow[r, "\phi"'] & (1, \Omega)
\end{tikzcd} \tag{6.3.4}$$

commutes. However, it is a pullback square if and only if $a = b$. This is basically due to Remark 6.3.7. Therefore, $\mathfrak{C}$ does not have a subobject classifier nor a power object since the characteristic maps $\phi : (b, Y) \to (1, \Omega)$ only represent conditional subsets of (b, Y) which live on b.

We have shown that the category of conditional sets has all limits and co-limits, a natural numbers object, is generated by the subobjects of its terminal object, and the natural partial order on the set of subobjects of a conditional set defines a complete Boolean algebra. However, it does not have a subobject classifier nor power objects. The category of conditional sets $\mathfrak{C}$ is a natural place for the concepts of conditional function, conditional inclusion, and conditional product, as defined in Chapter 3. More precisely, the objects of $\mathfrak{C}$ correspond to the class of conditional sets with respect to the relative algebras of some fixed $\mathcal{A} \in \mathscr{A}$. This subclass of all conditional sets is closed under the operations of conditional power set and conditional product, respectively, see the representation result in Proposition 6.2.2 and the Definitions 3.2.4 and 3.3.1. We have shown in the first three chapters of this thesis that it is possible to do mathematics on conditional sets, although we can not show that $\mathfrak{C}$ is a topos.

Appendix

1 Boolean algebras

The following definitions and results are based on [MKB89, GH09].

Definition 1.1 ([MKB89]). A *Boolean algebra* is a structure $(\mathcal{A}, \wedge, \vee, {}^c, 0, 1)$ with two binary operations $\vee$ and $\wedge$, a unary operation ^{c}, and two distinguished elements 1 and 0 such that for all a, b and c in $\mathcal{A}$,

(i) (commutativity) $a \wedge b = b \wedge a$ and $a \vee b = b \vee a$;

(ii) (associativity) $a \wedge (b \wedge c) = (a \wedge b) \wedge c$ and $a \vee (b \vee c) = (a \vee b) \vee c$;

(iii) (absorption) $(a \wedge b) \vee b = b$ and $(a \vee b) \wedge b = b$;

(iv) (distributivity) $(a \wedge b) \vee c = (a \vee c) \wedge (b \vee c)$ and $(a \vee b) \wedge c = (a \wedge c) \vee (b \wedge c)$;

(v) (complementation) $a \wedge a^c = 0$ and $a \vee a^c = 1$.

Let E be a any set and $\mathcal{P} = \mathcal{P}(E)$ its power set. The structure $(\mathcal{P}, \cap, \cup, {}^c, \emptyset, E)$ is a Boolean algebra, called the *power set algebra* of E.

On every Boolean algebra the relation $a \leq b$ if $a \wedge b = a$ is a *distributive complemented lattice*. More precisely, $\mathcal{A}$ has a least element 0 and a greatest element 1 such that for every $a \in \mathcal{A}$ there exists $b \in \mathcal{A}$ with $a \vee b = 1$ and $a \wedge b = 0$, where $\vee$ and $\wedge$ denote the supremum and the infimum of a and b, respectively, and it holds the distributive law $(a \wedge b) \vee c = (a \wedge c) \vee (b \wedge c)$. Boolean algebras can be described as being distributive complemented lattices, [MKB89, p. 16].

Definition 1.2 (Lemma and Definition 3.1, [MKB89]). Let $\mathcal{A}$ be a Boolean algebra and $a \in \mathcal{A}$. The set $\mathcal{A}_a = \{b \in \mathcal{A} : b \leq a\}$ is, with the partial order inherited from $\mathcal{A}$, a Boolean algebra, called the *relative algebra* of $\mathcal{A}$ with respect to a.

Definition 1.3 (Definition 1.2, [MKB89]). A *homomorphism* from a Boolean algebra $\mathcal{A}$ into a Boolean algebra $\mathcal{B}$ is a function $f : \mathcal{A} \to \mathcal{B}$ such that $f(0) = 0$ and $f(1) = 1$, and for all a, b in $\mathcal{A}$, it holds $f(a \vee b) = f(a) \vee f(b)$ and $f(a^c) = f(a)^c$. The mapping f is an *isomorphism* from $\mathcal{A}$ onto $\mathcal{B}$ if f is a bijective homomorphism. We say that $\mathcal{A}$ and $\mathcal{B}$ are *isomorphic* if there exist an isomorphism from $\mathcal{A}$ onto $\mathcal{B}$.

Definition 1.4 (Definition 1.28, [MKB89]). Let $\mathcal{A}$ be a Boolean algebra. The least upper bound (the greatest lower bound) of some $\mathcal{M} \subseteq \mathcal{A}$ in the partial order $(\mathcal{A}, \leq)$ is denoted by $\vee\mathcal{M}$ $(\wedge\mathcal{M})$, if it exists. $\mathcal{A}$ is *(σ-)complete* if $\vee\mathcal{M}$ and $\wedge\mathcal{M}$ exist for every $\mathcal{M} \subseteq \mathcal{A}$ (for every countable $\mathcal{M} \subseteq \mathcal{A}$).

By convention, $\vee_\emptyset := 0$ and $\wedge_\emptyset := 1$.

Definition 1.5 (Definition 5.1, [MKB89]). A homomorphism is (σ-)*complete* if it preserves $\vee\mathcal{M}$, in the sense of Definition 1.3, for every $\mathcal{M} \subseteq \mathcal{A}$ (for every countable $\mathcal{M} \subseteq \mathcal{A}$) for which $\vee\mathcal{M}$ happens to exist in $\mathcal{A}$.

Definition 1.6 (Definition 3.3 and 3.8, [MKB89]). Let a and b in $\mathcal{A}$ and $\mathcal{M} \subseteq \mathcal{A}$. We call a and b *disjoint* if $a \wedge b = 0$. $\mathcal{M}$ is a *pairwise disjoint family* if $a > 0$[1] for $a \in \mathcal{M}$ and any two distinct elements of $\mathcal{M}$ are disjoint. A Boolean algebra $\mathcal{A}$ satisfies the *countable chain condition* if each pairwise disjoint family in $\mathcal{A}$ is at most countable.

Theorem 1.7 ([Tar37, GH09]). *Every σ-complete Boolean algebra which satisfies the countable chain condition is complete.*

Definition 1.8 (Definition 2.3, [MKB89]). Let $\mathcal{A}$ be a Boolean algebra. An element $a \in \mathcal{A}$ is an *atom* of $\mathcal{A}$ if $0 < a$ but there is no $b \in \mathcal{A}$ such that $0 < b < a$. $\mathcal{A}$ is *atomless* if it has no atoms and *atomic* if for each positive element $b \in \mathcal{A}$, there is $a \in \mathcal{A}$ such that $a \leq b$.

Definition 1.9 (Definition 14.1, [MKB89]). Let κ and λ be two cardinals and $\mathcal{A}$ a Boolean algebra. $\mathcal{A}$ is (κ, λ)-*distributive* if for any sets I and J such that $|\, I \,| \leq \kappa$ and $|\, J \,| \leq \lambda$ and for any family $(a_{ij})_{i\in I,\, j\in J}$ in $\mathcal{A}$,

$$\bigwedge_{i\in I} \bigvee_{j\in J} a_{ij} = \bigvee \left\{ \bigwedge_{i\in I} a_{if(i)} : f \in J^I \right\},$$

where J^I is the set of all functions from I to J, provided that each of the unions (respectively intersections) in the previous formula exist in $\mathcal{A}$. $\mathcal{A}$ is (κ, ∞)-distributive if it is (κ, λ)-distributive for every cardinal λ. $\mathcal{A}$ is completely distributive if it is (κ, λ)-distributive for all cardinals κ and λ.

Theorem 1.10 (Theorem 14.5, [MKB89]). *A complete Boolean algebra is completely distributive if and only if it is atomic. In particular, a complete Boolean algebra is completely distributive if and only if it is isomorphic to a power set algebra.*

Definition 1.11 (Definition 5.17, [MKB89]). Let $\mathcal{A}$ be a Boolean algebra. A *congruence relation* on $\mathcal{A}$ is an equivalence relation $\sim$ on $\mathcal{A}$ such that, for all a, a', b, b' in $\mathcal{A}$, $a \sim a'$ and $b \sim b'$ imply $a^c \sim (a')^c$ and $a \vee b \sim a' \vee b'$.

Definition 1.12 (Definition and Lemma 5.18, [MKB89]). Let $\sim$ be a congruence on $\mathcal{A}$. For $a \in \mathcal{A}$, let $\pi(a) = \{a' \in \mathcal{A} : a \sim a'\}$ be the equivalence class of a with respect to $\sim$, and let $\mathcal{A}/_\sim = \{\pi(a) : a \in \mathcal{A}\}$ be the set of equivalence classes of $\sim$. There is a unique Boolean algebra structure on $\mathcal{A}/_\sim$ which makes $\pi : \mathcal{A} \to \mathcal{A}/_\sim$ an epimorphism of Boolean algebras. $\mathcal{A}/_\sim$ is the *quotient algebra* of $\mathcal{A}$ with respect to $\sim$; π is the *canonical homomorphism* form $\mathcal{A}$ into $\mathcal{A}/_\sim$. The congruence relation induced by π on $\mathcal{A}$ coincides with $\sim$.

[1] We write $a < b$ if $a \leq b$ and $a \neq b$.

Definition 1.13 (Definition and Lemma 5.19, [MKB89]). A subset of a Boolean algebra $\mathcal{I} \subseteq \mathcal{A}$ is an *ideal* of $\mathcal{A}$ if

(i) $0 \in \mathcal{I}$,

(ii) if $a \in \mathcal{I}, b \in \mathcal{A}$ and $b \leq a$, then $b \in \mathcal{I}$,

(iii) if $a \in \mathcal{I}$ and $b \in \mathcal{I}$, then $a \vee b \in \mathcal{I}$.

An ideal $\mathcal{I}$ is σ-complete if $\vee\mathcal{M} \in \mathcal{I}$ for each countable subset $\mathcal{M} \subseteq \mathcal{I}$.

Lemma 1.14 (Definition and Lemma 5.22, [MKB89]). *Let $\mathcal{I}$ be an ideal of $\mathcal{A}$. Then the relation $\sim$ on $\mathcal{A}$ defined by $a \sim b$ if and only if $(a \wedge b^c) \vee (a^c \wedge b) \in \mathcal{I}$ is a congruence relation on $\mathcal{A}$. The quotient algebra $\mathcal{A}/\sim$ is denoted by $\mathcal{A}/\mathcal{I}$. If $\mathcal{A}$ is a σ-complete Boolean algebra and $\mathcal{I}$ is a σ-complete ideal of $\mathcal{A}$, then the associated canonical epimorphism $\pi : \mathcal{A} \to \mathcal{A}/\mathcal{I}$ is σ-complete, and in this case, also $\mathcal{A}/\mathcal{I}$ is σ-complete.*

Definition 1.15 (Definition 1.1, Chapter 22, [MKB89]). A *measure algebra* is a pair $(\mathcal{A}, \mu)$ where $\mathcal{A}$ is a σ-complete Boolean algebra and $\mu : \mathcal{A} \to [0, \infty]$ is a function such that

(i) $\mu(a) = 0$ if and only if $a = 0$;

(ii) $\mu(\bigvee_{n\in\mathbb{N}} a_n) = \sum_{n\in\mathbb{N}} \mu(a_n)$ if $(a_n)_{n\in\mathbb{N}}$ is a sequence in $\mathcal{A}$ such that $a_m \wedge a_n = 0$ whenever $m \neq n$.

Let $(\Omega, \mathcal{F}, \mu)$ be any measure space. For $A \subseteq \Omega$, set

$$\mu^*(A) := \min\{\mu(B) : A \subseteq B \in \mathcal{F}\} \in [0, \infty].$$

The collection $N_\mu = \{A \subseteq \Omega : \mu^*(A) = 0\}$ is called the set of *negligible sets*. Note that $N_\mu \cap \mathcal{F}$ is a σ-complete ideal of $\mathcal{F}$. Let $\mathcal{A}$ be the Boolean algebra $\mathcal{F}/N_\mu \cap \mathcal{F}$ and $\pi : \mathcal{F} \to \mathcal{A}$ the canonical epimorphism. Then π is σ-complete, and there is a function $\nu : \mathcal{A} \to [0, \infty]$ defined by writing $\nu(\pi(A)) = \mu(A)$ for every $A \in \mathcal{F}$. Now $(\mathcal{A}, \nu)$ is a measure algebra, called the measure algebra *associated* to $(\Omega, \mathcal{F}, \mu)$, see [MKB89, p. 890-891].

It is well-known that generally $\mathcal{A}$ is not σ-isomorphic to a σ-algebra of sets (for example the associated measure algebra of the Borel space), see [Loo47].

Theorem 1.16 (Proposition 2.8, Theorem 2.10, Chapter 22, [MKB89]). *Let $(\Omega, \mathcal{F}, \mu)$ be a σ-finite measure space. Then the associated measure algebra $\mathcal{A}$ satisfies the countable chain condition and is (ω, ∞)-distributive.*

2 Toposes

The following definitions and results are based on [LM92, Lan98].

Definition 2.1 ([LM92]). A *category* $\mathcal{C}$ consists of a collection of *objects* and a collection of *morphisms* (or maps, or arrows) and four operations; two of these operations associate with each morphism f of $\mathcal{C}$ its *domain* $d_0(f)$ and its *codomain* $d_1(f)$, respectively, both of which are objects of $\mathcal{C}$. One writes $f : C \to D$ to indicate that f is a morphism of $\mathcal{C}$ with domain C and codomain D. The other two operations are an operation which associates with each object C of $\mathcal{C}$ a morphism 1_C (or id_C) of $\mathcal{C}$ called the *identity* morphism of C and a operation of *composition* which associates to any pair (f, g) of morphisms of $\mathcal{C}$ such that $d_0(f) = d_1(g)$ another morphism $f \circ g$, their composite. These operations are required to satisfy the following axioms:

(i) $d_0(1_C) = C = d_1(1_C)$;

(ii) $d_0(f \circ g) = d_0(g)$ and $d_1(f \circ g) = d_1(f)$ for any pair of maps (f, g) such that $d_0(f) = d_1(g)$;

(iii) $1_D \circ f = f$ and $f \circ 1_C = f$ for any two objects C and D of $\mathcal{C}$ and any morphism f from C to D;

(iv) $(f \circ g) \circ h = f \circ (g \circ h)$.

For two objects C and D, the collection of morphisms with domain C and codomain D is denoted by $\mathrm{Hom}(C, D)$. Sets and set functions form a category with the usual composition, denoted by **Sets**.

Definition 2.2 (Categorical Preliminaries, [LM92]). In an arbitrary category $\mathcal{C}$, a morphism $f : C \to D$ in $\mathcal{C}$ is called an *isomorphism* if there exists a morphism $g : D \to C$ such that $f \circ g = 1_D$ and $g \circ f = 1_C$. If such a morphism f exists, one says that C is isomorphic to D, and one writes $C \cong D$. A morphism $f : C \to D$ is called *epic* if for any object E and any two parallel morphisms $g, h : D \rightrightarrows E$ in $\mathcal{C}$, $g \circ f = h \circ f$ implies $g = h$. Dually, $f : C \to D$ is called a *monic* if for any object B and any two parallel morphisms $g, h : B \rightrightarrows C$ in $\mathcal{C}$, $f \circ g = f \circ h$ implies $g = h$; in this case, one writes $f : C \rightarrowtail D$.

Two monics $f : A \rightarrowtail D$ and $g : B \rightarrowtail D$ with a common codomain D are called *equivalent* if there exists an isomorphism $h : A \to B$ with $g \circ h = f$. A *subobject* of D is an equivalence class of monics into D. The collection $\mathrm{Sub}_{\mathcal{C}}(D) = \mathrm{Sub}(D)$ of subobjects of D carries a *natural partial order* defined by $[f] \leq [g]$ if and only if there is an $h : A \to B$ such that $f = g \circ h$, where $[f]$ and $[g]$ are the equivalence classes of $f : A \rightarrowtail D$ and $g : B \rightarrowtail D$.

In **Sets**, a monic is an injective, an epic a surjective function, the bijections are the isomorphisms and the collection of subobjects of a set X describes the set of subsets of X.

Definition 2.3 (p. 15, [Lan98]). A *subcategory* $\mathcal{D}$ of a category $\mathcal{C}$ is a collection of some of the objects and some of the arrows of $\mathcal{C}$, which includes with each arrow f both the object $d_0(f)$ and the object $d_1(f)$, with each object C its identity arrow 1_C and with each pair of composable arrows $C \to C' \to C''$ their composite.

Definition 2.4 (Categorical Preliminaries, [LM92]). Given a category $\mathcal{C}$, one can form a new category $\mathcal{C}^{\mathrm{op}}$, called the *opposite* or *dual category* of $\mathcal{C}$, by taking the same objects but reversing the direction of all the morphisms and the order of all compositions. In other words, an arrow $C \to D$ in $\mathcal{C}^{\mathrm{op}}$ is the same as an arrow $D \to C$ in $\mathcal{C}$.

Definition 2.5 (p. 51-52, [Lan98]). Let $\mathcal{C}$ be a category. A function R which assigns to each pair of objects A and B an equivalence relation $R_{A,B}$ on $\mathrm{Hom}(A, B)$ is a *congruence* on $\mathcal{C}$ if $f, f' : A \to B$ satisfies $(f, f') \in R_{A,B}$, then for all $g : A' \to A$ and all $h : B \to B'$ it holds $(h \circ f \circ g, h \circ f' \circ g) \in R_{A',B'}$. A congruence defines a new category, called the *quotient category* with respect to R, and is denoted by $\mathcal{C}/R$. Its objects are the objects of $\mathcal{C}$ and its hom-sets are the quotients $\mathrm{Hom}_{\mathcal{C}}(A, B)/_{R_{A,B}}$.

Definition 2.6 (p. 36, [Lan98]). For two given categories $\mathcal{C}$ and $\mathcal{D}$, one can construct a new category $\mathcal{C} \times \mathcal{D}$, called the *product* of $\mathcal{C}$ and $\mathcal{D}$. An object of $\mathcal{C} \times \mathcal{D}$ is a pair (C, D), where C is an object of $\mathcal{C}$ and D an object of $\mathcal{D}$, and an arrow $(C, D) \to (C', D')$ in $\mathcal{C} \times \mathcal{D}$ is a pair (f, g), where $f : C \to C'$ is an arrow in $\mathcal{C}$ and $g : D \to D'$ an arrow in $\mathcal{D}$. The composition of two arrows (f, g) and (f', g') is defined by $(f', g') \circ (f, g) = (f' \circ f, g' \circ g)$.

Definition 2.7 (Categorical Preliminaries, [LM92]). Given two categories $\mathcal{C}$ and $\mathcal{D}$, a *functor* from $\mathcal{C}$ to $\mathcal{D}$ is an operation F which assigns to each object C of $\mathcal{C}$ an object $F(C)$ of $\mathcal{D}$, and to each morphism f of $\mathcal{C}$ a morphism $F(f)$ of $\mathcal{D}$, in such a way that F respects the domain and codomain as well as the identities and the composition:

$$F(d_0(f)) = d_0(F(f)),\ F(d_1(f)) = d_1(F(f)),\ F(1_C) = 1_{F(C)},\ F(f \circ g) = F(f) \circ F(g),$$

whenever this makes sense. One writes $F : \mathcal{C} \to \mathcal{D}$. Let F and G be two functors from a category $\mathcal{C}$ to a category $\mathcal{D}$. A *natural transformation* α from F to G, written $\alpha : F \to G$, is an operation associating with each object C of $\mathcal{C}$ a morphism $\alpha_C : F(C) \to G(C)$ of $\mathcal{D}$, in such a way that for any morphism $f : C' \to C$ in $\mathcal{C}$, the diagram

$$\begin{tikzcd} F(C') \arrow[r, "\alpha_{C'}"] \arrow[d, "F(f)"'] & G(C') \arrow[d, "G(f)"] \\ F(C) \arrow[r, "\alpha_C"'] & G(C) \end{tikzcd}$$

commutes. The morphism α_C is called the *component* of α at C. If every component of α is an isomorphism, α is said to be a *natural isomorphism*. If $\alpha : F \to G$ and $\beta : G \to H$ are two natural transformations between functors $\mathcal{C} \to \mathcal{D}$, one can define a composite natural transformation $\beta \circ \alpha$ by setting $(\beta \circ \alpha)_C = \beta_{G(C)} \circ \alpha_C$. For fixed categories $\mathcal{C}$ and $\mathcal{D}$, this yields a new category $\mathcal{D}^{\mathcal{C}}$: the objects of $\mathcal{D}^{\mathcal{C}}$ are functors from $\mathcal{C}$ to $\mathcal{D}$, while the morphisms of $\mathcal{D}^{\mathcal{C}}$ are natural transformations between such functors. Categories so constructed are called *functor category*.

Definition 2.8 (Categorical Preliminaries, [LM92]). In an arbitrary category $\mathcal{C}$, an object C of $\mathcal{C}$ with the property that for any other object D of $\mathcal{C}$ there is one and only one morphism from D to C is called a *terminal object* of $\mathcal{C}$. An terminal object of $\mathcal{C}$, if it exists, is unique up to isomorphism and is usually denoted by 1. In any category with a terminal object 1 an arrow $x : 1 \to A$ is a *global element* of A. An object of C of $\mathcal{C}$ is an *initial object* of $\mathcal{C}$ if it is a terminal object in $\mathcal{C}^{\text{op}}$. It is often denoted by 1.

A singleton set $\{*\}$ is, up to isomorphism, the terminal, and the empty set the initial object in **Sets**.

Definition 2.9 (Categorical Preliminaries, [LM92]). An object X equipped with two morphisms $\pi_1 : X \to A$ and $\pi_2 : X \to B$ is a *product* of A and B if for any other object Y and any two maps $f : Y \to A$ and $g : Y \to B$ there is a unique map $h : Y \to X$ such that $\pi_1 \circ h = f$ and $\pi_2 \circ h = g$. This definition makes sense in any category and determines the object X (if it exists) to within isomorphism. It is common to denote a product of two objects A and B in an arbitrary category, if it exists, by $A \times B$.

In **Sets**, this definition matches the usual definition of the product of two sets.

Definition 2.10 (Categorical Preliminaries, [LM92]). In an arbitrary category $\mathcal{C}$, one says that a commutative square

$$\begin{tikzcd} P \arrow[r, "q"] \arrow[d, "p"'] & C \arrow[d, "g"] \\ B \arrow[r, "f"'] & A \end{tikzcd}$$

is a *pullback (square)*, if it has the property: given any object X of $\mathcal{C}$ and morphisms $\beta : X \to B$ and $\gamma : X \to C$ with $f \circ \beta = g \circ \gamma$, there is a unique $\delta : X \to P$ such that $p \circ \delta = \beta$ and $q \circ \delta = \gamma$. Given $f : B \to A$ and $g : C \to A$, the pullback P with its *projections* p and q is uniquely determined up to isomorphism (if it exists at all), and one usually writes $B \times_A C$.

In any category with terminal object, the pullback of $B \to 1$ and $C \to 1$ is the product $B \times C$, if it exists. The pullback of two functions $f : B \to A$ and $g : C \to A$ in **Sets** is the set

$$B \times_A C := \{(b, c) \in B \times C : f(b) = g(c)\}$$

together with the projections $\pi_1 : B \times_A C \to B$ and $\pi_2 : B \times_A C \to C$.

Definition 2.11 (Categorical Preliminaries, [LM92]). Given two morphisms $f : A \to B$ and $g : A \to C$ in $\mathcal{C}$, a diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] & B \arrow[d, "p"] \\ C \arrow[r, "q"'] & P \end{tikzcd}$$

is called a *pushout* if the corresponding diagram in $\mathcal{C}^{\mathrm{op}}$ is a pullback. The pushout of f and g, if it exists, is denoted by $P = B \coprod_A C$.

In **Sets**, the pushout of two functions $f : A \to B$ and $g : A \to C$ is the disjoint union $B \coprod_A C$ where two elements are identified if they have the same preimage in A. The mappings p and q are the respective embeddings. For the definition of finite limits and colimits we refer to [Lan98].

Definition 2.12 (IV.1, [LM92]). An (*elementary*) *topos* is a category $\mathcal{E}$ with

(i) A pullback for every diagram $X \to B \leftarrow Y$;

(ii) A terminal object 1;

(iii) An object Ω and a monic **true**:$1 \rightarrowtail \Omega$ such that for any monic $m : S \rightarrowtail B$ there is a unique arrow $\phi : B \to \Omega$ in $\mathcal{E}$ for which the following square is a pullback:

$$\begin{tikzcd} S \arrow[r] \arrow[d, "m"'] & \mathbf{1} \arrow[d, "\mathbf{true}"] \\ B \arrow[r, dashed, "\phi"'] & \Omega. \end{tikzcd}$$

In this case we call ϕ the characteristic map of m.

(iv) To each object B an object $P(B)$ and an arrow $\varepsilon_B : B \times P(B) \to \Omega$ such that for every arrow $f : B \times A \to \Omega$ there is a unique arrow $g : A \to P(B)$ for which the following diagram commutes:

$$\begin{tikzcd} B \times A \arrow[r, "f"] \arrow[d, dashed, "\mathrm{id}_X \times g"'] & \Omega \arrow[d, equal] \\ B \times P(B) \arrow[r, "\varepsilon_X"'] & \Omega. \end{tikzcd}$$

The object Ω is the *subobject classifier* of $\mathcal{E}$ and $P(B)$ the *power object* of B for each object B in $\mathcal{E}$.

Definition 2.13 (p. 50-51, [LM92]). A *Heyting algebra* H (or a Brouwerian lattice) is a bounded lattice, that is a lattice with a smallest element 0 and a greatest element

1, such that for all a and b in H there is a greatest element x of H such that $a \wedge x \leq b$. This element is the *relative pseudo-complement* of a with respect to b and is denoted by $a \to b$. The *pseudo-complement* of a is the relative pseudo-complement of a with respect to 1 and is denoted by $\neg a$.

The open sets of a topology $\mathfrak{T}$ on some set X together with the set inclusion form a Heyting algebra. The pseudo-complement of an open set O is given by $\neg O = \mathfrak{int}(O^c)$. Any complete and (infinitely) distributive lattice is a Heyting algebra. In particular, every complete Boolean algebra is a complete Heyting algebra.

For any object X in a topos $\mathcal{E}$, the poset $\mathrm{Sub}(X)$ of subobjects of X has the structure of a Heyting algebra, by [LM92, IV.8, Theorem 1]. A topos is *Boolean* if $\mathrm{Sub}(X)$ is in fact a Boolean algebra, see [LM92, VI.1, Proposition 1]. A topos $\mathcal{E}$ is *two-valued* if 0 and 1 are the only subobjects of the terminal object 1, see [LM92, p. 257]. A topos is *well-pointed*, if two parallel arrows $f, g : A \rightrightarrows B$ are equal if and only if $f \circ x = g \circ x$ for every global element $x : 1 \to A$, see [LM92, p. 236]. Well-pointedness is the expression of the fact that there is no distinction between global and local existence.

Proposition 2.14 (Proposition 7, VI.2, [LM92]). *A well-pointed topos $\mathcal{E}$ is both two-valued and Boolean.*

A family $\mathcal{G}$ of objects of a category $\mathcal{C}$ is said to *generate* $\mathcal{C}$ if and only if $f \neq g : A \to B$ in $\mathcal{C}$ implies that $f \circ u \neq g \circ u$ for some arrow $u : G \to A$ from an object G in the family $\mathcal{G}$, see [LM92, p. 275].

Proposition 2.15 (Proposition 8, Chapter VI.2, [LM92]). *Let $\mathcal{E}$ be a topos which is generated by subobjects of 1, and moreover has the property that for each object E, $Sub(E)$ is a complete Boolean algebra. Then $\mathcal{E}$ satisfies the axiom of choice.*

Definition 2.16 (p. 268-269, [LM92]). For an arbitrary topos $\mathcal{E}$, the axiom of infinity states that there is an object $\mathbf{N}$ of $\mathcal{E}$ with arrows

$$1 \xrightarrow{0} \mathbf{N} \xrightarrow{s} \mathbf{N}$$

such that for any object X of $\mathcal{E}$ with arrows x and f, as below, there is a unique arrow h which makes the following diagram commute:

$$\begin{tikzcd} 1 \arrow[r, "0"] \arrow[d, equal] & \mathbf{N} \arrow[r, "s"] \arrow[d, dashed, "h"] & \mathbf{N} \arrow[d, dashed, "h"] \\ 1 \arrow[r, "x"'] & X \arrow[r, "f"'] & X. \end{tikzcd}$$

The object $\mathbf{N}$ is then called a *natural numbers object* for $\mathcal{E}$.

Example 2.17. The category **Sets** is a well-pointed Boolean topos with a natural numbers object and satisfying the axiom of choice. Indeed, its terminal object is given by $1 = \{*\}$. Recall its pullbacks, as defined above. The subobject classifier is the set

$\Omega = \{0,1\}$, and the injection **true**:$1 \rightarrowtail \{0,1\}$ is given by $* \mapsto 0$. If $m : Y \rightarrowtail X$ is an injection for any set X, then the characteristic map $\phi : X \to \{0,1\}$ is the characteristic function of the image $m(Y) \sqsubseteq X$. The power object of any set X is the set $P(X) = \{\phi : X \to \{0,1\}\}$. The function $\varepsilon_X : X \times P(X) \to \{0,1\}$ is given by $(x,\phi) \mapsto \phi(x)$. For any function $f : X \times Y \to \{0,1\}$, describing a subset $A \subseteq X \times Y$, the unique map g is the characteristic function of the set $\{x \in X : (x,y) \in A\}$. The set of all natural numbers $\mathbb{N} = \{0,1,2,\ldots\}$ is the naturals numbers object, where the arrow 0 maps the single element of 1 to $0 \in \mathbb{N}$, while $s : \mathbb{N} \to \mathbb{N}$ is the usual successor function $n \mapsto n+1$. The category **Sets** is well-pointed, since two parallel functions $f, g : X \rightrightarrows Y$ are equal if and only if $f(x) = g(x)$ for all $x \in X$. Thus, **Sets** is Boolean and two-valued, by Proposition 2.14. The Boolean character of **Sets** also follows from the fact that the collection of subsets of every set is a complete Boolean algebra. Since any map $1 \to X$ describes an element of X, it follows that 1 generates the category **Sets**. Hence, in **Sets** the axiom of choice is satisfied, by Proposition 2.15. $\Diamond$

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Selbständigkeitserklärung

Ich erkläre, dass ich die vorliegende Arbeit selbständig und nur unter Verwendung der angegebenen Literatur und Hilfsmittel angefertigt habe.

Berlin, den 06.09.2013 Asgar Jamneshan